

Introduction

Algebraic thinking is a major area of school mathematics that is crucial for students to learn but challenging for teachers to teach.

—Maria Blanton, Linda Levi,
Terry Crites, and Barbara Dougherty (2011, p. 1)

Everywhere we turn these days, we encounter another article, report, or book on the importance of algebra. In 2000, with the publication of *Principles and Standards for School Mathematics*, the National Council of Teachers of Mathematics (NCTM) made algebra one of five mathematics content standards for preK–12 mathematics, describing it as a way of thinking that cuts across all math content areas and unifies the curriculum. In 2008, the final report of the president’s National Mathematics Advisory Panel identified algebra as a main concern “for Algebra¹ is a demonstrable gateway to later achievement” (p. xiii). In 2010, the National Governors Association Center for Best Practices and the Council of Chief State School Officers, building on the NCTM standards and heeding the Advisory Panel’s recommendations, echoed the importance of algebraic thinking starting in kindergarden, embodied in the Common Core State Standards for Mathematics published that year. Yet major publications continue to acknowledge a serious national mathematics problem and regularly identify algebra as the main culprit. Why? Because the mathematics achievement curve begins a sharp decline as students reach late middle school, precisely when U.S. students begin their study of algebra. Classic comments such as “I liked math until I began algebra” or “math made sense until algebra” add anecdotal evidence to scientific results.

To help remedy this situation, policy makers and math educators are calling for a strong foundation for algebra in the early grades in order to help students get a deeper understanding of mathematics early on and prepare them for success in algebra later on. But questions arise about how best to answer this call. After all, elementary teachers were not educated to teach algebra, and, what’s more, they may barely remember high school algebra. Some may even harbor unpleasant memories of their algebra experience.

Algebra in the early years, a relatively new focus area in mathematics education, has received much attention in the past decade or two. The movement, known as “early algebra,” is *not* about teaching traditional school algebra *early*. Rather, it’s about fostering ways of thinking, doing, and communicating about mathematics, and of teaching and learning mathematics with understanding. It’s about making connections, analyzing relationships, noticing structure, studying change, and solving problems; it’s about justifying, conjecturing, generalizing, symbolizing, and mathematizing, all of which are critical habits of mind for all of mathematics.

1. The word *Algebra* capitalized refers to a particular course or course sequence in middle or high school.

But this begs further questions: How can elementary teachers learn to cultivate these habits of mind? Where can they find the research on children’s algebraic thinking in forms that are clear and useful to them? In what ways can they use their present curriculum to meet algebra expectations? In short, what does it *mean* to plant the seeds of algebra in the elementary grades, and what do these seeds *look like* in grades 3–5? One answer is in your hands. *Planting the Seeds of Algebra, 3–5: Explorations for the Intermediate Grades (Planting Seeds)* is a pioneer in its genre: It takes the reader into real classrooms, describes students engaged in important mathematics, models teaching strategies, connects the early mathematics with advanced algebra concepts, and makes suggestions for further explorations. Based on existing research, these different components converge to offer meaningful algebraic experiences for young students.

To be more specific, I have written this book for teachers in the elementary grades to help instill in them a very different view of algebra than the popular one captured in Glasbergen’s cartoon below! It is my hope that readers will

- *Unlearn* any negative lessons about algebra they may be harboring and *release* any negative beliefs or attitudes that impede a full appreciation of the topic.
- *Experience* algebraic acculturation; that is, *cultivate* new thought and behavior patterns that naturally weave into algebra’s cognitive fabric.
- *Conceptualize* algebra as a domain that makes sense and connects to the world we live in, rather than as a set of meaningless symbolic formulas and procedures.
- *Understand* the continuum—and visualize meaningful bridges—between the mathematics they teach and the mathematics taught in secondary school algebra.
- *Visualize* concrete embodiments of what algebra actually looks like in the early grades, giving it color, light, and texture.
- *Enjoy* the usefulness, power, and beauty of algebra as an integral part of mathematics.



“Algebra will be useful to you later in life because it teaches you to shut up and accept things that seem pointless and stupid.”

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Why Write This Book?

I wish to articulate the hoped-for outcomes that are the motivating force behind this work as well as the anticipated macro-level consequences of reading and implementing the teachings within. There are six broad-ranging results to be gained from mathematical instruction that routinely “plants the seeds” of algebraic thinking.

Student Achievement and Teacher Empowerment. As teachers learn to “algebrafy” elementary mathematics by weaving into it a web of algebraic ideas and actions, words and

symbols, their students become better at doing mathematics, and together they enjoy the subject more. Teachers will be able to answer rather than dread *why* questions, such as, “Why invert and multiply when dividing fractions?” They will be motivated to pose deeper questions, and their students will gain deeper number sense, spatial sense, and symbol sense. When algebra becomes an organizing principle for elementary mathematics, the potential for increased math proficiency is huge. Mindful explanations engender profound understanding, which contributes to richer mathematical experiences now, which in turn lead to successful mathematics experiences later. After reading *Planting Seeds*, teachers will feel empowered, and students will feel *they can*.

Mindful explanations → Profound understanding → Richer experiences → Joy and success

Respect for Elementary Teachers. Throughout my 25 years of professional development with preK–14 teachers, I’ve heard high school teachers too often blaming elementary teachers for their students’ algebra ills. Granted, we have a national math problem, and algebra is clearly at center stage. But good will is not the missing ingredient: I’ve worked with many reflective, inquisitive, and assiduous elementary school teachers who are open minded and eager to learn.

The reasons for high school students’ poor preparation for algebra are at least twofold: (1) Elementary teacher preparation programs require little by way of mathematics, and (2) the elementary teachers who venture to take an algebra course are given traditional, advanced algebra material that doesn’t serve them directly in their elementary classrooms. Therefore, this book meets an urgent need: It provides elementary teachers with the tools to create a new classroom culture of algebraic approaches to mathematics, the fruits of which will have an upward domino effect, forcing secondary math teachers, in turn, to rethink their own algebra teaching practices. As a result of this inevitable change, high school teachers will stop trivializing the mathematics learned and taught in elementary school and begin valuing elementary teachers.

Lower School Math

The indoctrination begins. Students learn that mathematics is not something you do. . . . Emphasis is placed on sitting still, filling out worksheets, and following directions. Children are expected to master a complex set of algorithms . . . unrelated to any real desire or curiosity on their part. . . . Multiplication tables are stressed, as are parents, teachers, and the kids themselves.

—Paul Lockhart (2009, p. 134)

A Mathematics Revolution With Teachers as Agents of Change. There is a dire need to educate the public about algebra and mathematics in general; to reintroduce these noble intellectual achievements back into our culture as valuable assets; to change beliefs, attitudes, and behaviors. This can happen only through education. This education must start with the young. Elementary school teachers can be change agents in this burgeoning

revolution: They educate the leaders and decision makers of tomorrow. Only they can model for our children a mathematics that is vibrant, useful, exciting, and rich. Hence, this book will plant the seeds of a new awareness of the nature of algebra (and mathematics as a whole) that will begin to change current attitudes. It will spread from teachers to students and outward to society. A new awareness about the role of algebra in our world will gradually deviate from Opinion 1, below, and align itself with Opinion 2.

(1) Algebra isn't essential to much of anything. . . . It is useless torture. . . . It's for the few, not the many.

—Coleman McCarthy (1991, p. A21)

(2) Algebra represents one of mankind's great intellectual achievements—the use of symbols to capture abstractions and generalizations, and to provide analytic power over a wide range of situations, both pure and applied.

—Alan Schoenfeld (2008, p. 506)

The Democratization of Algebra. High school Algebra has served as a gatekeeper in education: It lets some people in to enjoy STEM careers (in science, technology, engineering, and/or mathematics) but keeps many out. I have been passionate about the need to widen algebra's narrow gate since the inception of my mathematics career. Still today, too few students pass Algebra with understanding. Since proficiency in algebra continues to be considered a criterion for success in higher education, failure in high school Algebra disadvantages a slice of the population that includes many female, Black, and Hispanic students, depriving them of scientific careers. Recent reports have further correlated success in higher education with completion of Algebra II. Reconceptualizing the nature of algebra in the early grades will provide *all* students opportunities to engage with big ideas in meaningful ways, thereby increasing the chances for more students to thrive in high school Algebra. This book will contribute to the early algebra reform efforts to transform Algebra from “a gateway for some to a highway for all” (Neagoy, 2010).

Solving the algebra problem serves four major goals . . . [including] to democratize access to powerful ideas by transforming algebra from an inadvertent engine of inequity to a deliberate engine of mathematical power.

—James Kaput (2008, p. 6)

Narrowing of the Global Achievement Gap. Beyond college and graduate school, we must also consider the quality of work and life itself. A growing number of educational scholars and leaders are concerned about our students' disadvantages on a larger scale. Harvard education professor Tony Wagner warns of a global achievement gap (Wagner, 2008). He argues that the 2001 No Child Left Behind Act, instead

of narrowing the achievement gap, has left us with ineffective schools that are unable to prepare our students for college, work, and life. With judicious reasoning, he makes the case that our students are unprepared to analyze arguments, weigh evidence, or detect biases. He draws a list of seven survival skills for today's teenagers, the core competencies deemed necessary for success in college and the workplace. Wagner's survival skills are strikingly similar to the habits of mind exhibited in classrooms where mathematics is taught and learned with meaning, as modeled in *Planting Seeds*. Four of them follow.

- Critical thinking and problem solving
- Agility and adaptability
- Effective oral and written communication
- Curiosity and imagination

Planting Seeds focuses on rethinking algebra as a network of ideas, actions, and symbols, and on fostering mathematical habits of mind that will serve students all the way through high school and beyond. These ways of thinking and doing will transfer to other areas of education and life and will be foundational to developing Wagner's survival skills.

One's intellectual and aesthetic life cannot be complete unless it includes an appreciation of the power and the beauty of mathematics.

—Jerry P. King (1992, p. 3)

A Love for Mathematics Akin to the Love for Language Arts. When it comes to mathematics, and especially algebra, we almost always justify its importance to students by portraying it as a prerequisite to (1) high school graduation, (2) college achievement, or (3) a lucrative career. A main ingredient is missing in this utilitarian portrayal of “one of humanity's most ancient and noble intellectual traditions” (RAND Mathematics Study Panel, 2003). The missing ingredient is a passion for mathematics, one akin to the delight in reading or the joy of writing. A deep appreciation is lacking for the intrinsic nature of mathematics: its art, elegance, and creativity. The many “wow moments” and “aha moments” teachers and students will experience when using explorations from *Planting Seeds* will begin to foster a profound enjoyment of mathematics that will lead to a contagious love for its awesome beauty.

I wish to emphasize the critical role schoolteachers play in kindling a child's love for mathematics. Their effect is critical not only on students' achievement but also on students' appreciation for the topic. It may be hard to prove, but those who were inspired by a teacher when they were young know the vital role the teacher played in their lives. Once students enter middle school, their minds are pretty much made up about mathematics. For the disheartened, there's a slim chance of any rekindling. We must therefore reach children while they are young and *turn their hearts and minds on* to mathematics. A child *turned off* to mathematics is a tragedy for the child and a tragedy for our country.

What Background Do I Bring to this Book?

First, *Planting Seeds* is the outgrowth of 25 years of working with teachers. I have worked specifically on algebra both in site-based contexts, where I taught professional development courses and workshops for teachers of all grade bands, in the United States and abroad; and at venues such as Georgetown University and the Carnegie Institution of Washington, where I designed, directed, and taught summer institutes for K–14 teachers.

Second, this book is based on the research knowledge acquired throughout my career, both through conducting and consuming research and especially through guiding large-scale, national algebra-related projects as a program director at the National Science Foundation under the umbrellas of the Teacher Professional Continuum (TPC) and the Centers for Learning and Teaching (CLT) programs.

Third, this book is also the fruit of my work over the past 18 years in creating, writing, and hosting television programs (e.g., on the Annenberg CPB Channel, www.learner.org) and video series (e.g., for Annenberg Media, Discovery Education, <http://store.discoveryeducation.com>, and T3 Europe, www.t3europe.org) on the teaching and learning of algebra in grades K through–14.

Last, this book draws on webinars, keynote speeches, and Math Show performances I've given on the topic of algebra at national and international conferences, cultural events, and back-to-school or math nights for parents. I am grateful to the many people who insisted that I "write a book on algebra that would inspire more people."

Who Will Benefit From This Book?

On a pragmatic level, I've written *Planting Seeds* primarily for preservice and in-service elementary school teachers, elementary math specialists, coaches, and teacher educators who work with elementary school teachers. Since more and more parents today are homeschooling their children, this book will also serve parents, as well administrators trying to make sense of the rapidly shifting K–12 algebraic landscape. On a philosophical level, this book joins my lifelong professional efforts to inspire teachers, students, and all lovers of learning by *infusing*, *injecting*, and *injecting* them with a fascination for the power, the value, and the beauty of mathematics.

I envision this book being used as a resource for the following:

- Courses for preservice teacher preparation and workshops, courses, and institutes for in-service teachers
- Site-based lesson study or other teacher collaborations
- Sessions for elementary teachers run by math specialists and coaches
- Math workshops for parents of young children
- Individual learning by teachers, administrators, and other readers interested in children's mathematical development and education

How Is This Book Organized?

At the heart of this book are three in-depth Explorations—I, *Circling Circles*; II, *Fancy Fences*; and III, *Multiplication Musings*. Explorations I, II, and III address core material for grades 3–5 students and are in turn composed of three chapters each:

1. *The Lesson* chapter (Chapters 1, 4, and 7) builds on actual lessons recorded in elementary classrooms, in public or private schools. Only the names of the students and teachers are changed.
2. The *Algebra Connections* chapter (Chapters 2, 5, and 8) revisits each stage of the exploratory lesson, bringing to light the often hidden connections between elementary mathematics and secondary algebra.
3. The *More Problems to Explore* chapter (Chapters 3, 6, and 9) begins with a *Next Steps* section offering suggestions for follow-up lessons. It also provides suggestions for further explorations for the teacher's own learning and ten additional explorations for the students' learning.

Chapter 10 describes a fourth exploration—IV, *Fractions in Action*, which is the fruit of professional development meetings with teachers, math lessons with students, and math night or math academy sessions with parents on fundamental aspects of fractions and rational numbers.

Finally, Chapter 11, titled *Final Thoughts*, summarizes the big ideas of the book—what algebraic thinking, talking, and doing means in grades 3–5—and offers six new “Monica’s Mottos,” maxims for teachers often inspired by them.

Exploration Topics

In characterizing “early algebra,” *Developing Essential Understanding of Algebraic Thinking for Teaching Mathematics in Grades 3–5* (NCTM, 2011) states, “Early algebra brings a more eclectic perspective to the kinds of activities that we might describe as algebra. . . . It offers multiple points of entry that draw on arithmetic, functional thinking, mathematical modeling, and quantitative reasoning” (p. 8). All four areas are addressed extensively throughout this book.

Exploration I looks at geometry, measurement, and data from an algebraic perspective. We learn that algebra stems from arithmetic and geometry and that, in fact, geometry and algebra are two complementary ways of thinking about mathematical ideas. Using modeling and quantitative reasoning, students discover a surprising property of a circle, the perfect Euclidean shape known since before the beginning of recorded history.

Exploration II focuses on geometric and numeric growing patterns and the not-so-obvious connections between patterns and functions and between functions and algebra. Again, through mathematical modeling and quantitative reasoning, the worlds of shapes, numbers, and algebra converge in meaningful ways. Students examine functions, or relationships between quantities that change, in two ways: recursively (looking at what changes) and rule-wise (looking at what remains the same or *invariant*).

Exploration III examines multiplication from multiple perspectives, beginning with images of multiplication children bring with them from their early experiences. Multiplicative thinking, a new big idea students encounter in grade 3, is the foundation for an entire network of interconnected concepts, including multiplication, division, fractions, ratios, rational numbers, proportional relationships, and linear functions—all of which are central to algebra. The connecting thread throughout Exploration III, and another big idea bridging basic computation to higher mathematics, is the distributive property of multiplication over addition. It is important for at least two reasons: It connects the two fundamental operations, and it helps students make sense of computational algorithms. Key aspects of division, such as partitive versus quotative, are also discussed.

Guided by eight challenging questions, such as, “Can $\frac{2}{5} + \frac{3}{4}$ ever equal $\frac{5}{9}$?” Exploration IV walks the reader through what is considered by many educators the most difficult topic for students to learn and teachers to teach: fractions and rational numbers. “Historically, the development of algebra grew from the arithmetic of fractions and rational numbers (Saul, 1998, p. 137).” Because they represent a big stumbling block for middle school students, solid groundwork must be laid in grades 3–5. Consequently, Exploration IV addresses five foundational aspects of fractions and rational numbers: meanings of $\frac{a}{b}$, representations of $\frac{a}{b}$, the whole or the unit, equivalence and comparison, and the meanings of division.

Grade Bands

Explorations I through IV are labeled Grade 3, Grades 4–5, Grades 3–4, and Grades 3–5 respectively. Nevertheless, with minor modifications, all explorations can be used in *any* of the three grade levels. Moreover, all elementary school teachers will find the wide range of algebraic connections and additional problems relevant to the mathematics they teach.

Common Core State Standards for Mathematics

The choices of what mathematical content and practices to include in this book—as well as what tasks, strategies, and approaches to teaching and assessing to include—were all made in light of the Common Core State Standards. This is true not only for the content standards linking operations and algebraic thinking but also for the standards on base-10 numbers, fractions, geometry, and measurement and data. If I had cited the pertinent CCSSM content or practice standard each time it was being modeled, there would have been too many Common Core references. Hence, I reserved citations for use only with the most challenging ones.

Fact: The standards do accommodate and prepare students for Algebra 1 in 8th grade by including the prerequisites for this course in grades K–7.

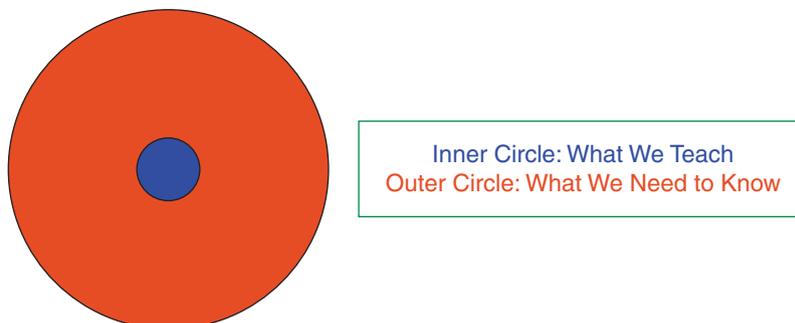
—www.corestandards.org/about-the-standards/myths-vs-facts

My Premises

As you read through and engage in this book’s explorations, you will no doubt notice the following premises, which have held true throughout my career:

Premise 1: We must desire to engage with mathematics. When mathematics is explored in ways that reveal its usefulness, its beauty, and its humanity, teachers and students alike develop a motivation and desire to engage with mathematics—on emotional and intellectual levels—two critical ingredients for a positive and rewarding mathematical experience.

Premise 2: We need to know much more than what we teach. Teachers are professionals. They understand that they must know much more than what they teach. I have shared this diagram with every teacher I've worked with as a representation of my strong belief in Premise 2.



A broad lens on learning enables teachers to build intelligently on students' existing foundations and erect new foundations for future study. A rich repertoire of knowledge empowers teachers to welcome rather than fear incisive questions, help *all* students grow in their knowledge, and inspire students with fascinating new mathematics.

Teachers with higher content knowledge are not only more apt to teach mathematics in a manner that is compatible with learning algebra but are also more inclined to hold their students to higher standards. . . . Not surprisingly, the quality of mathematics teaching depends on the teachers' knowledge of the content.

—Joy Darley and Barbara Leopard (2010, p. 185)

Premise 3: We must have a profound understanding of the mathematics we teach. The traditional U.S. curriculum has been characterized as “a mile wide and an inch deep,” because it traditionally attempted to “cover” so many topics while never “uncovering” any one topic in great depth. Consequently, an in-depth knowledge of the mathematics taught is often lacking. *Planting Seeds* helps remedy this situation.

Premise 4: If we are too comfortable, then we are not growing. If teachers become too comfortable with what they are teaching and how they teach it, they become set in their ways, and growth stagnates. Leaving the comfort zone now and then is positive and productive. Discomfort leads to change, change begets growth, and growth engenders new learning and excitement. As you delve into this book, you may feel a bit uncomfortable at times. Think of it as your own professional development, your adventure into new and stimulating mathematical territory, where one day you will be able to lead your students with confidence, mindfulness, and knowledge.

Premise 5: Raise the performance bar, and students will rise to meet you. “American students have not been succeeding in the mathematical part of their education at anything

like a level expected of an international leader” (National Mathematics Advisory Panel, 2008, p. 3). A main goal of the Common Core State Standards is to raise the level of mathematics education, described by some as two years behind international counterparts (Greenberg, 2008). I strongly believe *all* students should have the right to learn high-level mathematics. If you think the level of *Planting Seeds* is too advanced for your students, rather than worrying about what they *won't be able* to do, honor them with the belief that they *will be able* to think. When students sense your confidence in them, they develop self-confidence and rise to meet your expectations.

We have learned that children are capable of mathematical insights and mathematical invention that exceed our expectations. . . . The question of the expectations we knowingly or unknowingly set for our students is nowhere more crucial than in the gateway area called algebra.

—Joan Ferrini-Mundy, Glenda Lappan, and Elizabeth Phillips (1997, p. 282)

My Hope

For too many students, mathematics begins—in the world of arithmetic—as a meaningless set of numerical procedures, and then becomes—in the world of algebra—a meaningless set of symbolic procedures. It is my sincere hope that *Planting Seeds* will offer readers more breadth and depth in the mathematics they already know and new insights into algebra’s many faces: algebra as generalizing arithmetic; algebra as problem solving; algebra as examining structure; algebra as modeling real-world situations; and algebra the study of relationships between quantities that change. After reading this book, you will no longer think of algebra as *separate from* the mathematics you teach. This separation was typical of traditional curricula that long deprived students of deep mathematical thinking in the elementary grades and impeded them from learning algebra with understanding in the later grades. I trust you will see algebra not just as a garden of ideas but as a garden in full bloom awash with vibrant colors.

If I had to explain algebra to a student, I would say: “Think of all that you know about mathematics. Algebra is about making it richer, more connected, more general, and more explicit.”

—Ricardo Nemirovsky, as quoted by Erick Smith (2008, p. 133)