A Look at the Academic Research

Intervention in the Mathematics Classroom

INTRODUCTION TO THE ACADEMIC RESEARCH

This book provides a model for diagnosing errors involving fraction concepts/computation and providing *meaningful instructional strategies for timely, pinpointed intervention.* After this introduction to the research, the book begins with a two-part section titled "Big Ideas in Fractions and Problem Solving." That section is included because before students consider specific algorithms, they should have an understanding of basic fraction concepts—along with the types of *actions* and *problem structures* that are suggested by each operation. According to Huinker (2002), "Fundamental to operation sense is an understanding of the meanings and models of operations" (p. 72).

The units on fraction concepts and computation each begin with a Diagnostic Test (in multiple-choice format), and this is followed by an Item Analysis Table that keys incorrect test responses to specific error patterns. Each distractor on the tests is based on a specific error pattern. A comprehensive section, "Error Patterns & Intervention Activities," then follows. This section provides detailed analysis of error patterns with supporting Intervention Activities. The items used on the Diagnostic Tests are drawn from this section.

Lack of Conceptual Understanding—Error Patterns

"[Children frequently] either fail to grasp the concepts that underlie procedures or cannot connect the concepts to the procedures. Either way, children who lack such understanding frequently generate flawed procedures that result in systematic patterns of errors. . . . The errors are an opportunity in that their systematic quality points to the source of the problem and this indicates the specific misunderstanding that needs to be overcome."

—Siegler (2003, p. 291)

In her case study, Bray (2011) concluded that teachers "would benefit from a greater awareness of common student errors and how these errors are related to key mathematics concepts" (p. 35). Bray believes that teachers need support in developing teaching practices that use student errors in the classroom as springboards for class discussion.

Beattie and Algozzine (1982) note that when teachers use Diagnostic Tests to look for error patterns, "testing for teaching begins to evolve" (p. 47). And because diagnostic testing is just one of many tools to analyze student understanding, with each Item Analysis Table are additional suggestions to delve into the rationale of student errors.

According to Thanheiser (2009), "To help their students learn about numbers and algorithms, teachers need more than ability to perform algorithms. They need to be able to explain the mathematics underlying the algorithms in a way that will help children understand" (p. 277). Thames and Ball (2010) reported that "in recent studies of professional development, programs that connect the mathematical content to teaching practice produced greater gains in teachers' mathematical knowledge for teaching (subject matter knowledge combined with pedagogical content knowledge) and in their students' achievement than programs that focused merely on content knowledge" (p. 228).

Linking Research and Practice

"The call for a better linking of research and practice has been echoed in the mathematics education community for some time."

—Arbaugh et al. (2010, p. 4)

The Intervention Activities in this book are based on instructional practices supported by academic research that teach for *meaning* and *conceptual understanding*. The practices employed include accessing language, activating prior knowledge, using representations, using estimation and mental math, introducing alternative algorithms, participating in instructional games, and integrating technology. The research-based practices included in this book are written in an understandable style that should enable the typical teacher to implement the ideas in the classroom.

According to Kilpatrick, Swafford, and Bradford (2001), "When students practice procedures they do not understand, there is a danger they will practice incorrect procedures, thereby making it more difficult to learn correct ones. . . . Further, when students learn a procedure without understanding, they need extensive practice so as not to forget the steps" (pp. 122–123).

> "The Standards for Mathematical Content are a balanced combination of procedure and understanding."
>
> —*Common Core State Standards for Mathematics* (National Governors Association Center for Best Practices and Council of Chief State School Officers, 2010, p. 8)

A common addition error is shown in the sidebar. According to the National Council of Teachers of Mathematics (NCTM) (2000a), "Students who have a solid conceptual foundation in fractions should be less prone to committing computation errors than students who do not have such a foundation" (p. 218).

> **A Common Addition Error Pattern**
>
> $$\frac{2}{3} + \frac{3}{4} = \frac{5}{7}$$
>
> The student adds both the numerators and the denominators.

An important premise of this book is that when teachers analyze student work for conceptual and procedural misconceptions—and then provide timely, targeted, and meaningful intervention—the probability of the errors repeating in the future decreases. Hill, Ball, and Schilling (2008), citing the research of others, found that when teachers investigated how students learn particular subject matter, "their classroom practices changed and student learning was improved over that of teachers in comparison groups" (p. 376). According to Cox (1975), systematic errors (errors that occur in at least three out of five problems for a specific algorithmic computation) are potentially remediable, "but without proper instructional intervention the systematic errors will continue for long periods of time" (p. 152).

> "As we teach computation procedures, we need to remember that our students are not necessarily learning what we think we are teaching; we need to keep our eyes and ears open to find out what our students are *actually* learning. We need to be alert for error patterns!"
>
> —Ashlock (2010, p. 14)

It is important to emphasize that class or individual discussions of the errors should be conducted as part of a *positive* learning experience—one that allows for students to use reasoning and problem solving to explore why an erroneous procedure may not yield the correct answer.

Finally, any discussion on intervention would be incomplete without addressing key factors that affect the entire child, such as the principle of equity, student dispositions, and differentiating instruction. These areas are addressed in this research chapter.

EQUITY AND ACCESSING LANGUAGE

Equity in the mathematics classroom means providing all students with equitable, high-quality learning experiences.

> **The Equity Principle**
>
> "Excellence in mathematics education requires equity—high expectations and strong support for all students."
>
> —NCTM (2000a, p. 12)

Hiebert et al. (1997) define *equity* such that "every learner—bilingual students, handicapped students, students of all ethnic groups, students who live in poverty, girls, and boys—can learn mathematics with understanding. In order to do this, *each* student must have access to learning with understanding" (p. 65). The research of Campbell (1995) and others has shown that *all* children, including those who have been traditionally underserved, can learn mathematics when they have access to high-quality instruction and instructional materials that promote their learning.

A key to providing access to learning is to provide access to language. This book provides strategies to help math vocabulary terms become more accessible to English language learners and to the general student population through a feature titled "Accessing Language." In particular, words that have both conversational and mathematical meanings, and math terms that are often confused with one another, are addressed.

> *"Teachers must identify which words might cause confusion for their students and address these words during the lesson."*
>
> *—Cady, Hodges, and Brown (2010, p. 477)*

Since the passage of Public Law 94-142 in 1975 and its reauthorization as the Individuals with Disabilities Education Improvement Act in 2004, students with a variety of disabilities are increasingly being taught mathematics in inclusive classrooms. In fact, the majority of students with disabilities are now in regular classrooms for at least a portion of each school day. According to the work of Truelove, Holaway-Johnson, Leslie, and Smith (2007), when teachers implement instructional strategies designed to help those with learning disabilities, *all students*—not just those with disabilities—will likely benefit.

STUDENT DISPOSITIONS

> *"Students who have developed a productive disposition are confident in their knowledge and ability. They see that mathematics is both reasonable and intelligible and believe that, with appropriate effort and experience, they can learn."*
>
> *—Kilpatrick, Swafford, and Bradford (2001, p. 133)*

> *"When a child gives an incorrect answer, it is especially important for the teacher to assume that the child was engaged in meaningful activity. Thus, it is possible that the child will reflect on his or her solution attempt and evaluate it."*
>
> *—Yackel, Cobb, Wood, Wheatley, and Merkel (1990, p. 17)*

During the elementary grades, students often acquire individual views and dispositions toward the learning of mathematics that last for the rest of their lives. Such dispositions as curiosity, cooperation, and perseverance are personal habits that play a key role in future success with mathematics both in school and beyond.

An important question to ask is, "Why is it important to take student dispositions into account?" The answer may lie in the work of Dossey, Mullis, Lindquist, and Chambers (1988), based on various national assessments. They found that students who enjoy mathematics and perceive its relevance have higher proficiency scores than students with more negative perspectives. They also found that students become less positive about mathematics as they proceed through school; both confidence in and enjoyment of mathematics appear to decline as students progress from elementary to high school.

One implication of this research is that mathematics instruction should not only enable students to learn skills and understandings but also promote the *desire* to use what has been learned. According to Lannin, Arbaugh, Barker, and Townsend (2006), "Part of the process of learning and solving problems includes making errors that, if examined, can lead to further mathematical insight" (p. 182). Lannin et al., and others, believe that teachers should guide students to think and reflect about their errors through a process of recognizing, attributing, and reconciling.

This book—based on a philosophy of using error analysis with targeted interventions that are meaningful, along with follow-up instructional games and activities—is designed to promote *positive* learning experiences and favorable student dispositions toward mathematics.

Finally, children with emotional and behavioral disorders (EBD) often present a variety of challenges to educators. EBD students are especially prone to frustration when performing complex tasks. Guetzloe (2001) and others suggest that *nonaggressive* strategies be used with EBD students to encourage them to stay in class and in school.

> "If the student is misbehaving out of frustration with an activity, assisting the child with the activity will be more effective than punitive measures in correcting the behavior."
>
> —Truelove, Holaway-Johnson, Leslie, and Smith (2007, p. 339)

ACTIVATING PRIOR KNOWLEDGE

According to Steele (2002) and many others, teachers should review prerequisite skills or concepts no matter how long ago they were taught. Such review is even more important for students who have memory deficits, because they may quickly forget previously mastered skills, or they may have significant gaps in their knowledge. According to the TIMSS (Trends in International Math and Science Study), teachers in the United States tend to do most of the mental work of introducing, explaining, and demonstrating new concepts—and 60% of the time, they do not link new ideas with other concepts and activities. In Japan, where students scored near the top on the TIMSS, *teachers made explicit connections* in 96% of the lessons (U.S. Department of Education, 1996).

The Intervention Activities in this book build on students' prior knowledge by using familiar concepts and tools to develop new content. For example, students use their knowledge of number lines involving whole numbers to help them with number lines involving fractions. Overall, the strong conceptual development and connections among the various representations discussed in the opening unit on Fraction Concepts provides a solid foundation for later work with fraction operations.

> "One of the most reliable findings from research is that students learn when they are given opportunities to learn. Providing an opportunity to learn means setting up the conditions for learning that take into account students' entry knowledge, the nature and purpose of the tasks and activities, and so on."
>
> —Hiebert (2003, p. 10)

> **The Representation Standard**
>
> "Instructional programs from prekindergarten through grade 12 should enable all students to
>
> - create and use representations to organize, record, and communicate mathematical ideas;
> - select, apply, and translate among mathematical representations to solve problems;
> - use representations to model and interpret physical, social, and mathematical phenomena."
>
> —NCTM (2000a, p. 67)

REPRESENTATIONS

"The term representation refers both to process and to product" (NCTM, 2000a, p. 67). As a process, it refers to creating in one's mind a mental image of a mathematical idea. As a product, it refers to a physical form of that idea, such as a manipulative, an illustration, or even a symbolic expression. Why is the idea of representation so important? Simply stated, *the more ways a student can think about a mathematical concept, the better that student will understand the underlying mathematical idea.*

A Concrete → Semiconcrete → Abstract Model of Instruction: A number of studies suggest that concept development is strong when students begin with a tactile, hands-on model (concrete), move to the use of illustrations of those objects (semiconcrete), and finally move to a symbolic algorithm (abstract). Psychologist Jerome Bruner (1966) referred to those stages as *enactive, iconic,* and *symbolic.* Through his research, Bruner theorized that students learn mathematics better when their lessons progress through those three stages. Miller and Hudson (2007) found that such a three-stage model helps students with learning disabilities master concepts involving whole numbers, fractions, and algebra.

Many of the Intervention Activities in this book are designed so that students first encounter manipulatives, then refer to drawings of those objects, and finally develop computational proficiency by connecting those representations to an abstract algorithm. The National Library of Virtual Manipulatives (http://nlvm.usu.edu/en/nav/vlibrary.html) provides (for a fee) access to activities using virtual manipulatives. This provides the semiconcrete stage linking the concrete to the abstract.

Although the time spent developing concepts through the use of manipulatives and diagrams may be greater than the time needed to use a more traditional approach, less time is generally needed later for review and reteaching. Additional information on the use of representations is included in "The Teaching of Fractions" on pages 13–18.

Research: Hands-On Activities; Manipulatives; Diagrams

In a study of over 7,000 students, Wenglinsky (2000) found that students whose teachers conduct hands-on learning activities outperform their peers by more than 70% of a grade level in math on the National Assessment of Educational Progress (NAEP).

In a meta-analysis of 60 research studies, Sowell (1989) found that for students of all ages, math achievement is increased and students' attitudes toward math are improved with the long-term use of manipulative materials.

Ferrucci, Yeap, and Carter (2003) found, from their observations of Singapore schools and curricula, that modeling with diagrams is a powerful tool for children to use to enhance their problem-solving and algebraic reasoning skills.

Petit, Laird, and Marsden (2010), citing the research of others, point out that "students need to interact with multiple models that differ in perceptual features, which causes them to rethink and ultimately generalize the mathematical concepts being investigated with the models" (p. 5).

ESTIMATION AND MENTAL MATH

Traditionally, estimation and mental math have been thought of as supplemental skills. However, based on surveys of adults, Carlton (1980) found that most of the mathematics used in everyday living relies far more on estimation and mental computation than on traditional computation. Estimates, in particular, are popular because they are generally so much easier to work with than exact values—while providing results that are "close enough."

Also, traditionally, mental math and estimation have been taught *after* students master pencil-and-paper computation. However, Kilpatrick et al. (2001) found not only that children can learn to compute mentally and to estimate *before* learning formal pencil-and-paper computational procedures but also that mental math and estimation activities prior to formal work with computation actually enhance the learning of computation.

According to Van de Walle (2006), "The most important reference points or benchmarks for fractions are 0, $\frac{1}{2}$, and 1" (p. 303). Using benchmarks is an important strategy when estimating with fractions, because benchmarks provide "ballpark" numbers that are easy to work with. For example, benchmarks for $\frac{1}{12}$, $\frac{4}{9}$, and $\frac{7}{8}$ are 0, $\frac{1}{2}$, and 1, respectively. An estimate for the sum $\frac{1}{12} + \frac{4}{9} + \frac{7}{8}$ is $0 + \frac{1}{2} + 1$, or $1\frac{1}{2}$. Benchmarks can also be used to estimate with mixed numbers, as shown at the right. Here, an estimate is based on adding $4\frac{1}{2}$ and 3 because those numbers are close to $4\frac{8}{15}$ and $2\frac{9}{10}$, respectively.

Additional benchmarks that can be used (depending upon the ability levels of students) are $\frac{1}{4}$, $\frac{1}{3}$, $\frac{2}{3}$, and $\frac{3}{4}$. This book provides lessons that address the use of benchmarks for performing each of the operations for fractions and mixed numbers.

Reys and Yang (1998) conducted research among sixth- and eighth-grade students from several schools in Taiwan. In the study, the students were asked to perform the following calculation without using a calculator: $\frac{12}{13} + \frac{7}{8}$. They were also asked to perform the "parallel" estimation problem shown at the right. In the study, 61% of the sixth graders and 63% of the eighth graders obtained the correct result for the exact calculation. The results for the related estimation problem (shown at the right) reveal that about half of the sixth graders (52%) and about a fourth of the eighth graders

Estimation involves a process of obtaining an approximate answer (rather than an exact answer).

Mental math involves a process of obtaining an exact answer in your head.

"When students have regular opportunities to estimate, share orally, evaluate, compare their approaches, and transfer strategies to new settings, they feel challenged and, ultimately, empowered."

—Rubenstein (2001, p. 443)

Using Benchmarks to Estimate the Sum of Mixed Numbers

$$4\frac{8}{15} + 2\frac{9}{10} \longrightarrow 4\frac{1}{2} + 3 = 7\frac{1}{2}$$

Research Item (Reys & Yang, 1998)

Without calculating an exact answer, circle the best estimate for $\frac{12}{13} + \frac{7}{8}$.

A. 1 **B.** 2 **C.** 19 **D.** 21 **E.** I don't know

Results: Percents of Students Who Selected Each Choice

	6th Grade	8th Grade
A. 1	10%	20%
B. 2*	25%	38%
C. 19	36%	14%
D. 21	16%	12%
E. DK	13%	16%

(*) denotes the correct response.

> "Although exact answers are important, it is even more important that students develop a sense of reasonableness when working with numbers."
>
> —Reys (1998, p. 112)

(26%) chose 19 or 21 as the best estimate. The research showed little evidence of the use of benchmarks in obtaining the estimates. Clearly, students who recognize that both fractions are close to 1 should conclude that the best estimate is 2. According to Reys (1998), "This sharp difference in performance level when students computed exact answers and when they estimated answers suggests that high levels of computational proficiency do not guarantee that students have an equivalent level of number sense" (p. 112).

Research: Asking Children to Compare Estimation Strategies

Star, Kenyon, Joiner, and Rittle-Johnson (2010), citing several research studies, concluded that "a promising approach that has emerged from research in mathematics education and cognitive psychology emphasizes the role of comparison—comparing and contrasting multiple solution methods—in helping students learn to estimate" (p. 557).

Research: Children's Thinking on Mental Math

Many students think that mental math is nothing more than doing a traditional algorithm in your head. Reys and Barger (1994) found that teaching and practicing the written algorithms before doing any mental math actually increases the likelihood that children will think that way.

ALTERNATIVE ALGORITHMS

An *algorithm* is "a precise, systematic method for solving a class of problems" (Maurer, 1998, p. 21). In school mathematics, students generally learn a traditional algorithm for each operation that is quite efficient. Although many students experience success using traditional algorithms, some students do not.

> **History of the Word *Algorithm***
>
> Around 780–850 CE, Muhammad ibn-Musa al-Khwarizmi wrote *Book on Addition and Subtraction After the Method of the Indians* (title translated from the Arabic). In his book, solutions to problems are given in steps, or recipes. The word for these recipes, *algorithm*, is derived from the Latin that begins with *Dixit Algorismi*, or "al-Khwarizmi says."
>
> —Pickreign and Rogers (2006, pp. 42–47)

Unfortunately, some teachers give struggling students *more* instruction and practice using the same algorithms for which those students have already demonstrated failure. According to Ellis and Yeh (2008), the traditional algorithms "are very efficient but not very transparent—they do not allow students to see why the methods work. When students learn traditional algorithms by rote, they often come to think of this as *the* way to do arithmetic rather than as *one* way among many" (p. 368).

These students often continue to struggle with the following kinds of questions:

- When you add fractions, why do the denominators have to be the same? And why do you add the numerators, but not the denominators?

- When you multiply fractions, why do you multiply both the numerators and the denominators? And why is the product of two proper fractions *smaller* than either of the two fractions?
- When you divide fractions, why do you invert the divisor and multiply? (Some people recite the rhyme: "Ours is not to reason why; just invert and multiply.") And why is the quotient sometimes *greater* than the dividend when dividing with fractions?

This book provides extensive, step-by-step Intervention Activities to address the traditional algorithms. The Intervention Activities also frequently include *alternative algorithms*. According to Lin (2007/2008), alternative methods help students "understand how other algorithms work and prompt them to think more deeply about numbers and equations" (p. 298). It should be noted that alternative algorithms not only are effective with students who struggle with traditional algorithms but are also effective with *all* students up front—and may be used *instead* of those algorithms (or in addition to them). Many textbook programs include alternative algorithms with their materials because they benefit *all* students.

DIFFERENTIATING INSTRUCTION

According to Stiff, Johnson, and Johnson (1993), "If all students were the same, a teacher's job would be simple—and boring. Researchers would develop one comprehensive theory of learning; teachers would simply follow the recipe to produce high levels of success for 'all' students. The challenge is to find the combination of strategies that enable all students to reach their full potential" (p. 12).

One way to differentiate instruction is to use *scaffolding*. The Intervention Activities in this book are presented through *step-by-step instruction* with *guided questions* to pose to students—thus providing effective scaffolding.

According to Martin (2006), "To meet the needs of all students and design programs that are responsive to their intellectual strengths and personal interests, we must explore alternatives to traditional mathematics instruction. We need to examine not only what is taught, but how it is taught and how students learn" (p. iv).

Tomlinson (1999) advocates that teachers make accommodations to *content* (what you want students to learn), *process* (the way students make sense out of the content),

> "The depressing thing about arithmetic badly taught is that it destroys a child's intellect and, to some extent, his integrity. Before they are taught arithmetic, children will not give their assent to utter nonsense; afterwards they will. Instead of looking at things and thinking about them, they will make wild guesses in the hopes of pleasing the teacher."
>
> —Sawyer (1943)

> "The standard algorithms used in the United States are not universal. . . . As our schools become more and more diverse, it is important that students' knowledge from their home cultures is valued within the classroom. Having students share alternative methods for doing arithmetic is one way to do so and honors the knowledge of their parents and community elders."
>
> —Ellis and Yeh (2008, p. 368)

> **More Than One Way to Perform an Operation**
>
> "Most people have been taught only one way, so they quite naturally assume that there is only one way. The realization that there are many possible procedures to follow when operating on numbers can change the way that people think of mathematics."
>
> —Sgroi (1998, p. 81)

> **Scaffolding**
>
> Scaffolding refers to assistance provided to students (temporary supports that are gradually removed) to allow them to engage at a higher level than they would be able to without the assistance. Kilpatrick et al. (2001) concluded that by offering a subtle hint, posing a similar problem, or asking for ideas, students are assisted in their ability to reason.

One Size Doesn't Fit All

"The idea of differentiating instruction to accommodate the different ways that students learn involves a hefty dose of common sense, as well as sturdy support in the theory and research of education."

—Tomlinson and Allan (2000)

Gardner's Multiple Intelligences

- Verbal/Linguistic
- Mathematical/Logical
- Visual/Spatial
- Musical/Rhythmic
- Bodily/Kinesthetic
- Interpersonal
- Intrapersonal
- Naturalistic

Student Writing

"By writing we find out what we know, what we think. Writing is an extremely efficient way of gaining access to that knowledge that we cannot explore directly."

—Smith (1982, p. 33)

Interactive Instruments at the LdPride Web Site

- Find out your dominant intelligence.
- Find out your learning style.

(See www.LdPride.net; there may be a fee to obtain the test results.)

and *product* (student outcomes at the end of the lesson), because, as she is known for saying, "one size does not fit all." According to Pierce and Adams (2005), differentiating instruction involves first determining which of those parts of the lesson you want to tier. This decision is based on students' readiness and learning styles. According to Little, Hauser, and Corbishley (2009), "Through tiering, mathematics teachers can give all students challenging tasks while ensuring sufficient scaffolding for struggling students and reducing repetition for more advanced students" (p. 36).

Cognitive research on multiple intelligences (Gardner, 1991) provides strong evidence demonstrating the need for children to experience a variety of pedagogical methods. Gardner concluded that students "possess different kinds of minds and therefore learn, remember, perform, and understand in different ways" (p. 11). As Martin (2007) puts it, "Consider trying to learn to dance by reading a book and memorizing the steps. We learn when we are actively involved in the learning process and use a variety of learning modalities. Not all students have the same talents, learn the same way, or have the same interests and abilities" (p. iv).

Through the use of questioning, mathematical reasoning, and representations, this book tiers "process" while addressing the verbal/linguistic, mathematical/logical, and visual/spatial intelligences outlined by Gardner.

The verbal/linguistic intelligence is also addressed through student writing. This book provides suggestions for students to (1) write about procedures used in algorithms, (2) compare different algorithms for a given operation, and (3) write word problems that can be solved using a given operation. The writing may be viewed as a form of alternative assessment—providing a way for teachers to tier "product." According to Fello and Paquette (2009), "Writing in mathematics classrooms is imperative for students to describe their thinking processes, their methodology for solving problems, and their explanations for solutions" (p. 413). See page 17 for additional information on problem solving and problem formulation.

According to the LdPride website, www.LdPride.net, "Information about learning styles and multiple intelligences is helpful for everyone, but especially for people with learning disabilities and attention deficit disorder. Knowing your learning style will help you develop coping strategies to compensate for your weaknesses and capitalize on your strengths" (para. 1).

INSTRUCTIONAL GAMES AND TECHNOLOGY

A number of studies suggest that the use of instructional games has the simultaneous goals of improved learning outcomes and increased student motivation for learning mathematics. Good instructional games provide "authentic" experiences for the construction and reinforcement of concepts—while ensuring that every child has an opportunity to participate. Holton, Ahmed, Williams, and Hill (2001) reported that it is often difficult to convince students to check their answers: "In the context of a game, however, checking conjectures has a clear purpose—if the conjecture is wrong, then the child is likely to lose. In this regard, games provide an opportunity for teachers to question students about their thinking. One of the inhibiting factors in learning new concepts is the fear of failure and of getting wrong answers. Incorrect strategies within game situations are not recorded for later correction and so the stigma of failure does not exist" (p. 406). Of course, later review and discussion of effective and ineffective game strategies is encouraged to bring out the rich learning that can occur with games.

A growing body of research provides evidence that technology can play a positive role in academic achievement. According to Brooks-Young (2010), "One way to move education forward is to embrace emerging technologies that make it possible to implement programs where students master core academic content, hone applied 21st-century skills, and learn how to find success in an increasingly digital world" (p. 3). A list of suggested online resources (for instructional games, animated activities, and more), organized according to the major units of this book, is provided in the section "Technology Resources Online" on pages 225–227.

A key goal of this book is to provide material for teachers to use to make their math classes more meaningful and engaging. The instructional games, along with the other activities in the book, are designed to serve that purpose.

> "When teachers use appropriate mathematics games, both student learning and motivation are strengthened. . . . Mathematics games can and should be used before, during, and after instruction to help students develop higher-level thinking skills. . . . Games can stimulate children to be alert, curious, and critical, and to see themselves as problem solvers."
>
> —Thornton and Wilson (1993, pp. 288–289)

> **Communicating About Mathematics Using Games**
>
> "Mathematical games can foster mathematical communication as students explain and justify their moves to one another. In addition, games can motivate students and engage them in thinking about and applying concepts and skills."
>
> —NCTM (2000b)

> "Research has shown that whiteboards have a positive impact on student motivation and learning. The versatility of the whiteboard encourages its use with all types of learners."
>
> —Wolf, Lindeman, Wolf, and Dunnerstick (2011, pp. 557–558)

Research: Instructional Games in Mathematics

Klein and Freitag (1991) found that the use of instructional games increases student interest, satisfaction, and continuing motivation.

Research: Use of Technology

Sivin-Kachala and Bialo (2000) reviewed 311 research studies and concluded that technology-rich environments promote increased achievement in preschool through high school for both regular and special-needs students in all subject areas, improved attitudes toward learning, and increased student self-esteem.

RESPONSE TO INTERVENTION

Response to Intervention (RTI) is a multitier approach to the early identification and support of students with learning needs. Rather than testing students for learning disabilities after achievement failure has occurred, RTI identifies students whose performance does not match that of their peers early in the learning process so that they can receive assistance before they fall behind. RTI provides appropriate, increasingly intense research-based interventions to match each student's needs. Core features include differentiated instruction, guided and independent practice, and frequent progress monitoring with data-driven decision making (the use of student-performance data to continually evaluate the effectiveness of teaching and to make more informed instructional decisions).

RTI frequently is implemented as a three-tiered model. The three tiers generally used are similar to those described at left. Students who are participating in intervention programs at *any* of these tiers are part of the target audience for the instructional strategies provided in this book. For this reason, the material in this book may be delivered to the full classroom, to small groups, or to individuals.

De Corte, Greer, and Verschaffel (1996) found that learning is enhanced when teachers have access to the knowledge that learners bring to the lesson, use this knowledge as part of instruction, monitor students' changing conceptions as the lesson proceeds, and provide appropriate intervening instruction. According to Safer and Fleischman (2005), "Research has demonstrated that when teachers use student progress monitoring, students learn more, teacher decision making improves, and students become more aware of their own performance" (p. 82).

According to Fisher and Kopenski (2007/2008), the use of *item analysis* is an effective way for teachers to diagnose student misconceptions, to improve and adjust instruction, and to prevent or reduce errors. Teachers using this book should find the error-analysis approach to be a valuable progress-monitoring tool. The sets of practice exercises at the end of each unit provide an additional progress-monitoring

Tier 1: Universal Interventions

Universal interventions occur in the classroom for all students; they are preventative, proactive, and differentiated.

Tier 2: Targeted Group Interventions

Targeted group interventions provide additional interventions to Tier 1 instruction for at-risk students who demonstrate a specific need; they involve frequent assessment.

Tier 3: Intensive, Individual (or Small Group) Interventions

Intensive, individual (or small group) interventions are administered; they are assessment based, of high intensity, and of longer duration.

—Adapted from Batsche et al. (2005)

tool, because they may be used as posttests (due to the fact that they are broken into parts that align with the parts of the Diagnostic Tests).

Meaningful practice is another aspect of RTI. Sutton and Krueger (2002) found that sufficient practice is essential for learning mathematics, but it is also essential that students *understand* the skill being practiced—so that they do not inadvertently practice incorrect procedures.

This book builds on the evidence cited in this chapter on academic research by providing a model of assessment or diagnosis (that is manageable and ongoing), Intervention Activities (delivered early in the process and that teach for understanding with multiple approaches), and practice or follow-up activities—enabling teachers to use real-time data to meet the needs of individual students.

Web Sites That Provide Information and Resources About RTI

- National Center on Response to Intervention: www.rti4success.org
- RTI Action Network: www.rtinetwork.org

"Without information about [their] students' skills, understanding, and individual approaches to mathematics, teachers have nothing to guide their work."

—Mokros, Russell, and Economopoulos (1995, p. 84)

"We teach children to look for patterns in dealing with numbers; these patterns help children discover the structure of our number system. Similarly, teachers must look for patterns in the data they collect from children who are experiencing problems in computational skills. Recognizing patterns in the errors a child is making—that the child is, in other words, making a systematic error—is the initial step toward remediation of the error."

—Cox (1975, p. 156)

THE TEACHING OF FRACTIONS

According to Van de Walle (2006), "Fractions have always represented a considerable challenge for students, even into the middle grades" (p. 293). Studies have concluded that students find fraction concepts to be much more difficult than whole number concepts. For example, comparing and ordering whole numbers is relatively easy; you can find the next number by inspection. But with fractions, the notion of "next" number does not apply (because between any two fractions there is always another fraction). Skypek (1984) concluded, "We need a method more powerful than inspection to determine the *order* of rational numbers" (p. 12).

The ability to work with fractions is also extremely important. The ability to understand and compute with them is a skill needed in most walks of life. Further, in terms of what is needed by middle grade students for success in the learning of algebra, the National Mathematics Advisory Panel (2008) states that it is "the most important foundational skill not presently developed" (p. 18).

To help improve the instruction and learning of fractions, this author examined the research and literature to identify best practices. Some of those

"One of the most well established facts in all mathematics education literature is that performance on fractions is undesirably low."

—Hope and Owens (1987, p. 37)

"Using fractions did not come naturally to the students."

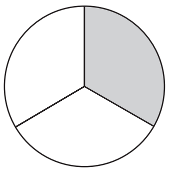
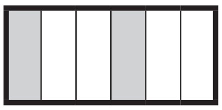
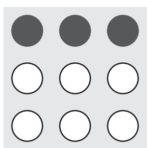
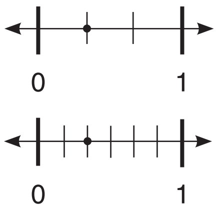
—Johanning (2008, p. 304), research conclusion drawn about middle school students' fraction literacy practices are described in this section—and also appear in the Intervention Activities throughout the book.

In a study involving more than 1,600 fourth and fifth graders, Cramer, Post, and delMas (2002) found that students engaged in a program for initial fraction learning that emphasizes the use of and translation among multiple modes of representation—pictorial, manipulative, verbal, real-world, and symbolic—had statistically higher mean scores (on concepts, order, transfer, and estimation) on a posttest and retention test than students in a control group using a regular commercial program. Citing the summaries of several studies, Cramer et al. (2002) state that students' difficulties with learning about fractions "are related in part to teaching practices that emphasize syntactic knowledge (rules) over semantic knowledge (meaning) and discourage children from spontaneous attempts to make sense of rational numbers" (p. 112). They strongly believe that "conceptual understanding should be developed before computational fluency" (p. 112).

> "Symbolic operations can become the focus of instruction once students have developed coherent and stable meanings that they may express symbolically."
>
> —Thompson and Saldanha (2003, p. 109)

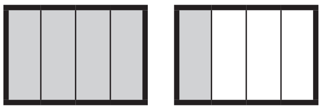
According to Kilpatrick et al. (2001), "Understanding a mathematical idea thoroughly requires that several possible representations be available to allow a choice of those most useful for solving a particular problem" (p. 102). A fraction may represent part of a unit whole that has been divided into equal parts (using a circle or a length model), part of a collection of objects, a location on a number line, or the quotient of two integers. Those four types of models, shown below for the fraction $\frac{1}{3}$, are used throughout this book.

> "It is important that they [children] be able to translate . . . between symbolic representations and the number line or pictorial representations."
>
> —Kilpatrick, Swafford, and Bradford (2001, p. 102)

Part-Whole		***Parts of a Collection***	***Location on a Number Line***	***Quotients of Integers***
Circle Model	**Length Model**	**Set Model**	**Measurement**	**Symbolic**
				$\frac{1}{3} = 1 \div 3 = 0.\overline{3}$ $\frac{3}{9} = 3 \div 9 = 0.\overline{3}$
Of 3 equal parts, 1 is shaded.	Of 6 equal parts, 2 are shaded. This is equivalent to "of 3 equal parts, 1 is shaded."	Of 9 objects, 3 are shaded. This is equivalent to "of 3 rows, 1 is shaded."	$\frac{1}{3}$ and $\frac{2}{6}$ show the same location, or distance from 0, along the number line.	$\frac{1}{3}$ and $\frac{3}{9}$ name the same number.

Although there are pros and cons to using each model—and some are more difficult than others for students to comprehend (as described below)—all students should gain experience with each model.

Part-Whole (Circle or Length Model): With the part-whole model, students use spatial skills to see how a fraction is related to the unit whole. In this respect, the circle model is more efficient than the length model, because students can usually relate sectors of a circle to the whole. With the part-whole model, students are also able to compare and find equivalent fractions. In this respect, the length model is usually more efficient than the circle model, because the length models can be more easily aligned to show equivalence. Izsák, Tillema, and Tunç-Pekkan (2008), citing the research of others, note that when using the length model for fractions greater than 1, some students were unable to focus on a fixed unit whole and thus "misinterpreted two units, each divided into four pieces, as one unit divided in eight pieces" (p. 36).



Some students may erroneously conclude that $\frac{5}{8}$ is shaded rather than $1\frac{1}{4}$.

Parts of a Collection (Set Model): This model has the advantage of allowing students to be able to manipulate physical objects (such as chips) to represent concepts. The model lends itself nicely in making strong connections to the many real-world situations where parts of collections involving discrete objects are considered. In particular, as described at the end of this section, this model is an ideal vehicle for making connections to probability.

Number Line (Measurement Model): On a number line, a fraction represents a *measure* of a quantity, where the interval from 0 to 1 is *one whole.* The model helps students connect fractions to measures on a ruler (partitioned, say, in parts of an inch), thus providing a context for students to see fractions as numbers. According to Lamon (2005), "When we talk about rational numbers as measures, the focus is on successively partitioning the unit" (p. 170). These measurements are called points on the number line.

Students use a number line by viewing fractions and mixed numbers as distances of points from 0. Parallel number lines can be used to show equivalence and make comparisons. A number line can also be used to locate fractions between any two given fractions (*Density Property of Rational Numbers*), because the unit whole can be continually broken down into any number of subdivisions.

The *Common Core State Standards for Mathematics* call for the use of number lines in the instruction of fraction concepts beginning at Grade 3. This book embeds the use of number lines throughout the Intervention Activities.

As a cautionary note, Bright, Behr, Post, and Wachsmuth (1988) and others found that many students experience difficulties working with number lines. Bright et al. concluded, "Since the model consists of pictorial information with

> "It is unlikely that any other interpretation of rational number comes close to the power of the number line for building number sense."
>
> —Lamon (2005, p. 173)

> **Common Core State Standards (Grade 3)**
>
> "Understand a fraction as a number on the number line; represent fractions on a number line diagram.
>
> Represent a fraction 1/*b* on a number line diagram by defining the interval from 0 to 1 as the whole and partitioning it into *b* equal parts. Recognize that each part has size 1/*b* and that the endpoint of the part based at 0 locates the number 1/*b* on the number line."
>
> —National Governors Association Center for Best Practices and Council of Chief State School Officers (2010, p. 24)

accompanying symbols, there may be a difficulty in connecting the information contained in the two modes of representation" (p. 16).

> "The realization that fractions represent division and constitute the most common way in which division is represented in algebra has caused a demand for increasing competence with fractions by all those for whom algebra skills are important."
>
> —Usiskin (2007, p. 370)

Quotient of Integers (Symbolic Model): Clarke, Roche, and Mitchell (2008), citing earlier research, state, "The notion of 'fraction as division' is not a common construct in most people's minds. If we understand, for example, that one meaning of $\frac{2}{3}$ is '2 divided by 3,' then strategies in sharing situations become obvious quite quickly" (p. 377). Note that this model is abstract if students perform the indicated division without really understanding why they are dividing. But the model makes strong connections between fractions and decimals, and the conversions allow for easy comparisons and ordering. Knowledge that a fraction can be interpreted as an indicated division is an important skill throughout mathematics, especially in "sharing situations" (such as finding how much pizza each person gets when 3 pizzas are shared equally among 5 people). See Problem 8 on page 20 for suggestions on introducing fractions to young children using equal sharing problems. Such problems provide a natural, concrete way for students to understand why a fraction is the quotient of the numerator by the denominator.

> "The evidence shows that learners use different cognitive skills in dealing with the various interpretations of rational numbers."
>
> —Driscoll (1984, p. 34)

Sequencing of the Models: Students should have experiences using all of the models that have been described in this section. Payne (1984) found that whereas 98% of third graders could shade $\frac{3}{4}$ of a unit whole, only 31% could locate $\frac{3}{4}$ on a number line. He also found that with the "parts of a collection" model, students often compared the shaded objects to the objects not shaded (rather than to the total number of objects in the set). Payne concluded, "After children understand the model using regions, they can be taught to relate this meaning to distance on a number line and to sets of objects" (p. 15).

> "In a guided-discovery lesson, students sequentially uncover layers of mathematical information one step at a time and learn new mathematics. . . . We cannot expect students to have a sustained interest in mathematics if formulas are served up like fast food."
>
> — Gerver and Sgroi (2003, p. 6)

Algorithms: This book provides hands-on lessons (some involving guided discovery) to introduce each of the operations. These lessons are intended for all students—not just for those in intervention situations. Driscoll (1984), citing a summary of a series of studies on fractions, noted that "the use of concrete objects that fit well with steps in algorithms appeared to help achievement, and this was more pronounced in retention." The Intervention Activities in this book provide such step-by-step, hands-on work.

The importance of teaching the algorithms for understanding is highlighted by the research of Prediger (2008). In a study of 269 students in Grades 7 and 9, only 43% correctly answered Item 2 in the sidebar on page 17. (When two proper fractions are multiplied, the product is less than the two fractions; when two improper fractions are multiplied, the product is greater than or equal to the two fractions.) Of the 109 students who chose incorrect choice **a.,** 88% also chose a wrong answer to Item 9 (most chose incorrect choice **b.**). On the other hand, of the 43% who answered Item 2 correctly, only 53% chose an incorrect answer for Item 9. Thus, understanding what happens when two fractions are multiplied is essential for understanding the concept of finding a fraction *of* another number.

Problem Solving and Problem Formulation: Lappan and Bouck (1998) discuss the importance of teaching fraction computation through problem solving. According to them, "The contexts of the problems help students make sense of how to put fractions together and take them apart" (p. 184). Prediger (2008), citing several empirical studies related to students' difficulties with fractions, states that "whereas *algorithmic competencies* are usually fairly developed, *understanding* is often weaker, as well as the *competencies to solve* word problems" (p. 29).

Problem writing is an effective vehicle for open-ended assessment. Although it may involve student formulation of just a single problem, the product usually reveals much more about student understanding than the solving of a word problem that often involves not much more than the use of an algorithm. This book provides models (problem structures) for each operation along with opportunities for students (and teachers) to formulate and solve word problems for each operation.

Writing problems involving fractions is not easy. Prediger (2008) found that students were much more successful in writing an adequate word problem for the addition of fractions than for the multiplication of fractions. He found that whereas 78% of 269 students could write a word problem for $\frac{2}{3} + \frac{1}{6} = \frac{5}{6}$, only 11% could write an adequate one for $\frac{2}{3} \cdot \frac{1}{4} = \frac{2}{12}$. Citing earlier findings of Fischbein, Deri, Sainati Nello, and Sciolis Marino (1985), Prediger concurred that "one difficulty lies in the mathematical fact that the most dominant mental model, the repeated addition, cannot be continued for $\frac{2}{3} \cdot \frac{1}{4} = \frac{2}{12}$" (p. 35). Difficulties with writing problems involving the division of fractions are discussed on page 156. Sample word problems involving fractions for each of the operations are provided in the section "Actions and Operations" on pages 24–32.

Finally, this author strongly believes that the formulation of problems provides a wonderful opportunity for students to make strides in achieving the level of expertise described in these standards.

Research Conducted by Prediger (2008)*

Item 2

Which statement is correct?
When I multiply two fractions,

a. the solution is always bigger than the two fractions.

b. the solution is always smaller than the two fractions.

c. the solution is sometimes bigger, sometimes smaller than the two fractions.**

Item 9

How can we calculate $\frac{2}{3}$ of 36?

a. $36 - \frac{2}{3}$ **b.** $36 \div \frac{2}{3}$ **c.** $\frac{2}{3} \cdot 36$**

d. none of these, but this:

*The 269 students in this study attended German grammar schools with a student body representing the highest achieving 40% of students in Germany.

**correct answer

"Problem writing provides an easy-to-implement assessment that readily reveals students' understandings and misunderstandings."

—Barlow and Drake (2008, p. 331)

Common Core State Standards

Standards for Mathematical Practice

Make sense of problems and persevere in solving them.

"Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution."

—National Governors Association Center for Best Practices and Council of Chief State School Officers (2010, p. 6)

Connections to Probability: Probability sometimes receives minimal coverage in the classroom due to time constraints, the fact that it may occur as the "last chapter in the book," or perhaps due to its perceived "lack of importance" at a particular grade level. This author concurs with NCTM that it is important for students to learn about probability at all grades. To facilitate instruction in probability, this book provides suggestions and materials for *integrating* key probability concepts into the curriculum as an *application of fractions*. It should be noted that the *Common Core State Standards* do not address key probability concepts until Grade 7. School districts that desire to have some probability topics taught earlier than that should find these integration ideas useful.

As shown in the sidebar, connections can be made to probability when teaching fractions, especially when teaching the "parts of a collection" model. Just as the entire collection of objects is considered to be one whole when working with fractions, in probability the sum of all possible (nonoverlapping) events is also one whole, or 1. This book makes connections to probability via the following four mini-lessons:

- Probability, Part 1: Introduction to Probability (pages 49–50)
- Probability, Part 2: Adding Probabilities (page 107)
- Probability, Part 3: Prediction (pages 127–128)
- Probability, Part 4: Probability of Successive Events (pages 140–142)

Spangler (2011), via activity lessons designed for midle school students, makes connections between probability and geometry, measurement, ratios/proportions, weather forecasting, music, and lotteries.

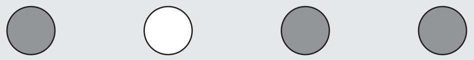
Probability

"Instructional programs from prekindergarten through grade 12 should enable all students to—

- understand and apply basic concepts of probability."

—NCTM (2000, p. 48)

Connections to Probability



You select one chip without looking. The probability of selecting a shaded chip is $\frac{3}{4}$. The probability of selecting a chip that is *not* shaded is $\frac{1}{4}$. The probability of selecting a chip that is either shaded or *not* shaded is $\frac{3}{4} + \frac{1}{4} = \frac{4}{4} = 1$.

"A subject in its own right, probability is connected to other areas of mathematics, especially number and geometry."

—NCTM (2000, p. 51)

"The CBMS (Conference Board of Mathematical Sciences) specifically notes that strengthening rational number knowledge and rational number sense is absolutely essential in the preparation of middle grade mathematics teachers. It further indicates that besides being able to explain procedures, these future teachers need a sufficient depth of understanding to be able to write problems that require specific arithmetic operations. Prospective teachers who obtain this deep understanding of rational numbers will be more prepared to help their future students to develop their own understandings."

—Carbone and Eaton (2008, p. 39)

QUESTIONS FOR TEACHER REFLECTION

1. Explain why it is so important to provide "access to language" in the mathematics classroom.

2. Although estimation with whole numbers is widely covered in the mathematics classroom, estimation with fractions is often barely covered. Why do you suppose that is the case? Why is it important for students to learn how to estimate with fractions?

3. Meyer (2010), in a talk titled "Math Class Needs a Makeover" (on TEDTalks at www.ted.com/talks/dan_meyer_math_curriculum_makeover.html), said the following about his situation as a mathematics teacher:

   > "I sell a product to a market that doesn't want it, but is forced by law to buy it."

   Describe some instructional practices that teachers can implement in their classrooms that would encourage more students to become "buyers" of mathematics. Your response may include any of the areas discussed in this research section along with any outside sources or experiences.

4. Some educators believe that learning how to use an alternative algorithm for an operation is a means in and of itself—and that students who learn an alternative algorithm need not master the traditional algorithm for that operation. Other educators believe that teaching an alternative algorithm is a means for moving students to ultimately use the traditional algorithm. Describe your position on this issue, and support it with examples and/or personal experiences.

5. Explain why it is important for students to gain experience with each type of fraction model listed below. Discuss some unique features of each model.

   Part-whole (length and circle models)

   Parts of a collection model

   Number line model

   Quotient of integers model

6. Consider the research by Prediger (2008) cited on pages 16–17.

   a. For each of the following situations, provide two numerical examples with answers:

      the product of two proper fractions,

      the product of a proper fraction and a whole number,

      the product of an improper fraction and a whole number, and

      the product of two improper fractions.

b. Under which situation(s) above is the product less than both original numbers? When is the product greater than or equal to both original numbers?

c. How is knowledge of the way numbers behave helpful in performing operations?

7. Why should teachers consider integrating probability concepts into their lesson plans when they teach fractions?

8. Empson and Levi (2011) argue that a productive way to introduce and develop fractions in all of the elementary grades is through the use of equal sharing problems such as the following:

> "4 children want to share 10 submarine sandwiches so that everyone gets the same amount. How much can each child get?" (p. 29)

According to Empson and Levi (2011), "Equal sharing problems allow your students to learn fractions using what they already understand as a foundation" (p. 28). Empson and Levi acknowledge that introducing fractions through a problem whose solution is a mixed number goes against conventional practice, but they believe that "when children solve problems that involve a set of objects that they can count and individually split into parts, it helps them understand that a countable set of objects can also include fractions of an object" (p. 6). They posit that the early use of equal sharing problems helps students "avoid the misconceptions that fractions are not numbers at all or appear only between 0 and 1" (p. 6).

a. Describe some possible strategies students might use to solve the above problem. Include diagrams with your discussion and the grade level(s) of the students you have in mind.

b. Describe some of the advantages and disadvantages of introducing fractions to young children via equal sharing problems versus another method.