1

An Introduction to Problem Solving

Problem solving has been an integral part of mathematics throughout recorded history. Over the past half century, there have been many vacillating agendas that have driven state curricula and ultimately the instruction of mathematics throughout the United States. Although many books have been written to address this all-important aspect of mathematics instruction over the past decades, the topic was not formally addressed until 1977, when the National Council of Supervisors of Mathematics (NCSM) pointed out that "learning to solve problems is the principal reason for studying mathematics" (NCSM 1977, 1). This marked the beginning of the problem-solving movement in school mathematics in the United States. More recently, the National Council of Teachers of Mathematics (NCTM), in their *Professional Standards for Teaching Mathematics,* stated that "Problem solving, reasoning and communication are processes that should pervade all mathematics instruction and should be modeled by teachers" (NCTM 1991, 95). Even earlier, in 1980, in its *Agenda for Action,* the NCTM stated that "Problem solving must be the focus of the curriculum" (NCTM 1980, 1). These comments are as valid today as they were then. In fact, what has changed is the approach to problem solving, not only as a separate topic of instruction, but also as a vehicle for teaching the skills and concepts of mathematics.

Today, problem solving is no longer merely another movement among the many that have appeared and disappeared in mathematics education. Rather, problem solving has now been accepted by most teachers as an integral part of their mathematics curriculum that must be taught alongside the arithmetic skills that are so necessary for success in school and in real life afterwards. In fact, problem solving provides the reason for teaching the skills of arithmetic.

Before we discuss problem solving, let's examine just what we mean by a "problem." A problem is a situation that confronts the learner, that requires resolution, and for which the path to the answer is not immediately known. It is this very definition of a problem that reduces many of the "word problems" teachers do from "problems" to mere "exercises." Teachers often group problems by types, and demonstrate to the class how to approach them. These might include problems dealing with uniform motion, age, mixture, percent, and so on. Usually, students are shown how to do one of these problems and told that the others are very similar and should be done the same way, albeit with different numbers! These we shall refer to as "exercises" rather than problems, because recognition of the type of problem immediately provides the learner with the path (or method) for arriving at the correct answer. Little thinking on the part of the students is required; rather, all they need do is recognize the type of problem and recall what the teacher's approach had been. Unfortunately, if the specific type of problem has not been taught, the children are often baffled, because they have not learned the new and different type of exercise. Although the problems in this book have been grouped by strategy for convenience, one should take note that once a strategy has been learned, it can then be applied to a variety of situations. Strategies are generic; they can be applied widely—exercise "types" cannot.

Now that we know what constitutes a problem, let's look at problem solving. Problem solving can be thought of in several different ways. First of all, problem solving may be considered a topic of instruction. That is, problem solving is a subject in the mathematics curriculum that must be taught to the children in the same way that multiplication, long division, percents, and so on, are taught. It can't be learned incidentally; it must be stressed and carefully taught! Second, problem solving may also be considered a mode of instruction. We can teach our mathematics classes using problem solving as the underlying thread to unite all the mathematics we teach. Problem solving provides a rationale for teaching the skills of arithmetic. Finally, problem solving is a way of thinking. That is, students cannot expect to learn to be problem solvers without careful structure of the process. Although some students intuitively may be good problem solvers, most of our students must be taught how to think, how to reason, and how to problem solve.

The way students approach problems will vary from child to child. However, one thing is clear. They will most often approach a problem

based upon their backgrounds and experiences. This can range from recognizing a problem as similar to one previously seen in class, or to taking on homework exercises similar to those discussed in class that day. In most cases, the student is not doing any problem solving, any reasoning, or any thinking. The student is simply mimicking (or copying) the skills learned earlier in class. If they do not recognize the type of problem (exercise), they may just sit back and be baffled. Students must be given the appropriate background, instruction, and support if they are to become effective problem solvers. They must learn how to solve problems and practice this sometimes elusive skill.

It is our goal in this book to make teachers aware of the ways that different problem-solving strategies can be used to provide elegant and efficient solutions to problems. Sometimes, teachers unwittingly convey to their students that there is one and only one way to solve a problem. The teacher may discourage the students from trying different strategies to solve problems, or may insist upon a unique strategy when approaching a problem. If the children are to become problem solvers, then the classroom must be a nonthreatening environment. Students must be free to express themselves and to try what they think is the way they wish to solve the problem, even if the approach leads to a dead end. The answer is not as important as the solution process used to obtain that answer.

One fact that teachers comment on, over and over, is the lack of a single resource containing sufficient problems to enable them to teach the strategies of problem solving. In fact, some teachers at the upper elementary school level are not even aware of the strategies that exist. They may recognize a few strategies from the textbook they use in class, but they are not aware of the existence of some of the other strategies that will be discussed in this book. The book is designed for the upper elementary school teacher who has a sincere desire to help his or her students succeed as problem solvers. It is divided up into chapters, each devoted to a specific strategy widely used in problem solving. The strategies are presented and applied to many problems so that you can more easily present the strategies to your students. Select the problems that are appropriate for your students' age and mathematical maturity. Use them as vehicles to teach the strategies of problem solving in your mathematics class. They are an excellent supplement to the problems and strategies that you will find in your textbook.

PROBLEM SOLVING ON ASSESSMENT TESTS

Most states are now requiring students to solve problems on the state mathematics tests. In addition to testing the children's abilities to master the basic skills of arithmetic, the test makers are placing a great emphasis on problem solving. Open-ended problems that do not have a

single definite answer challenge the children to adapt their own knowledge and experiences to help solve the problems. Rubrics are designed to accurately score the student responses and provide guidance to teachers to help students become better reasoners, thinkers, and problem solvers. The students are confronted with problems that require careful thought and reasoning, but also a knowledge of the strategies to be used to solve the problems. There is, additionally, a heavy emphasis on having the students write a careful explanation of what they did and why they did it.

THE HEURISTICS OF PROBLEM SOLVING

In 1945, George Polya published a book titled *How to Solve It.* This book was the forerunner of the problem-solving movement long before it actually started. In this book, Polya discussed the use of "heuristics" as a plan for solving problems. Heuristics is the process by which a problem solver attempts various approaches to find the solution to a problem. His heuristic model contained four steps. The student must first *read* the problem and think about it. He or she must carefully identify what information is given and what is to be found. Excess information is eliminated. Next, the student decides upon a *plan.* A strategy is suggested to be used to solve the problem. In the third stage, the student applies the strategy that was selected and tries to solve the problem, so as to arrive at the correct answer. In the fourth and final step, the student *looks back* at his or her solution and answer to make certain that his or her work is correct and that the question asked has, indeed, been answered correctly. It is this four-step plan that is the basis for the heuristic plans in most textbooks today, regardless of the names given to the steps or even the number of steps suggested.

Notice that a heuristic plan is quite different from an algorithm. If a student recognizes the appropriate algorithm and applies it correctly, he or she is guaranteed to arrive at the correct answer. Because a heuristic plan is only a model or guide, it can be applied correctly without guaranteeing the correct answer. However, the heuristic model does provide a guide for solving problems in general.

HOW TO USE THIS BOOK

This book has been designed as a resource for the teacher of the upper elementary school grades (3–6). Problem solving should play a major role in all mathematics instruction. When we teach children to be problem solvers, we are teaching them to think and to reason, skills that will be critical for

their entire lives—even beyond mathematics! And, if we are to succeed in teaching them to think, they need something about which to think. Problems provide this "something." We know that some teachers in the elementary school do not have an adequate source of problems to illustrate each of the problem-solving strategies. In most cases, you are limited to the problems that are provided in the class textbook. If students do not learn a particular strategy on the first or second attempt, the opportunity to reteach the strategy may not occur again in the book. We hope this book will remedy this dilemma.

Begin by reading through this book. Each chapter begins with a problem—posed for the teacher—that is representative of the particular strategy to be discussed. Take your time working out the problem, and then compare your solution with that suggested. Take some time to work out several of the problems intended for the students. Then compare your solutions with those suggested. Read the section titled *Teaching Notes.* When you feel comfortable, use these problems to teach this strategy to your students. Teachers typically learn a subject thoroughly when they teach it! Be ready to accept any suggestions made by the children. Carry out their suggestions to see if they provide an alternate solution to the problem, or merely lead to a dead end. In either case, valuable mathematics will be learned by the children. Notice that we refer to "answer" and "solution" as different. The *solution* is the entire process from the original encounter with the problem to the very end. The *answer* is the end product of the solution. As we have stated before, the answer must be correct, but we must put more importance on the process that was used to arrive at the answer.

PROBLEM DECKS

You should begin to create a math-problem set of cards (say, 5" × 9"). Every time you encounter a math problem that is appropriate for your students, write it on a card together with its solution and answer. On the back of the card, you might put the strategy or strategies used to solve the problem, together with the mathematics topic needed to solve the problem, or where it might fit in your curriculum. Feel free to copy problems directly from this book; they will provide an excellent starter set of problems. Keep adding to your set of cards whenever you encounter a problem you feel is appropriate for your students.

Some teachers may prefer to develop their problem deck on their computer. That is, they set up a series of folders of strategies, and each time place the problem where it is most appropriate. Provide the same information on this electronic math-problem set as you would on the cards. In any case, whenever you need a problem, take it from your deck and use it with the students.

THE STRATEGIES OF PROBLEM SOLVING

We have divided the rest of the book into nine strategies. Each chapter is devoted to an individual strategy, with many problems designed to illustrate how the strategy is applied. Here is the list of strategies:

- Organizing the Data
  - Creating a List
  - Making a Table
- Intelligent Guessing and Testing
- Solving a Simpler, Equivalent Problem
- Acting It Out/Simulating the Action
- Working Backwards
- Finding a Pattern
- Logical Reasoning
- Making a Drawing
- Adopting a Different Point of View

Your own textbook may have some or all of these strategies included. Not all of the strategies are appropriate for every grade or even for every student within a given grade. You must decide what is appropriate for your students and act accordingly. You might decide to modify a particular problem by making the numbers simpler or asking for only part of the problem. This is up to you—no one knows your students better than you do!