WHAT YOUR COLLEAGUES ARE SAYING . . .

"A great resource to encourage the most important goal of all—students doing the mathematical thinking!"

Jo Boaler
Nomellini-Olivier Professor of Mathematics Education, Stanford University
Stanford, CA

"If you love *Building Thinking Classrooms in Mathematics* but want more examples of tasks you can use with your middle and high school students, this is the book you've been waiting for. It goes deeply into fifty tasks, how to launch them, the task sequence, how to consolidate them, what student notes might look like, check-your-understanding questions you can use, as well as notes from Peter Liljedahl and Kyle Webb. It's a wonderful compliment to the main BTC book."

Robert Kaplinsky
Educational consultant and Co-founder, Open Middle
Lakewood, CA

"Peter Liljedahl's work continues to be a game-changer in mathematics education. With co-author Kyle Webb, this expansion of *Building Thinking Classrooms in Mathematics* to include manipulatives and technology isn't just a welcome addition, it's an equity move that empowers students to express their thinking and be recognized as capable, insightful problem-solvers."

Pamela Seda
Founder & CEO, Seda Educational Consulting, LLC
Co-author, *Choosing to See: A Framework for Equity in the Math Classroom*
Atlanta, GA

"This book doesn't just help you choose better tasks—it helps you create better math experiences. For students with math trauma, that's the first step towards rebuilding confidence, curiosity, and trust in the classroom!"

Vanessa "The Math Guru" Vakharia
Author, *Math Therapy*™
Toronto, Ontario, Canada

"Peter Liljedahl and Kyle Webb deliver what middle and high school math teachers are craving: tangible ways to make complex topics accessible, stimulate deep thinking and discussions, and supercharge learning for each student. These tried-and-true tasks, along with updated BTC practices for both die-hards and newcomers alike. They not only provide teachers with something awesome to try right away, but also valuable skills to design their own thinking tasks."

Jason To
K–12 Mathematics Coordinator
Toronto, Ontario, Canada

"Always practical and rooted in real classrooms, this follow-up builds on the original book *Building Thinking Classrooms in Mathematics* with research updates and thinking tasks already tried with students. Each task is fully supported—from launch to consolidation to check-your-understanding questions. More than a resource, this is a transformative guide that centers access and equips you for the flow, challenge, and joy of getting students thinking."

Fawn Nguyen
Director of STEM Initiatives, Amplify
Oak View, CA

"This is a game changer for grades 6 to 12 teachers and students! Blending the latest research with fifty ready-to-use lessons, it supports engagement, thinking, and deep learning of curricular outcomes, and it shows you how to design your own. A must-read for anyone building thinking classrooms—practical tools, deep thinking, and joyful learning all in one book."

Alicia Burdess
Numeracy Lead Teacher, Grande Prairie and District Catholic Schools
Grande Prairie, Alberta, Canada

"I cannot stress enough how incredibly useful this book is in supporting my work with students and teachers. The wide array of rich tasks gives me lots of options and allows me to focus my attention on planning and preparing the learning experiences rather than creating or searching for highly engaging thinking tasks. This is a must-have resource for all middle and high-school math teachers and providers of professional development."

Mike Flynn
CEO, Flynn Educational Consulting
Northampton, MA

"Peter Liljedahl has once again given us a book that is as practical as it is transformative. With his trademark clarity and vision, he shows secondary math teachers how to create classrooms where all students can engage deeply and meaningfully in mathematics. A must-read for anyone committed to reimagining math teaching in ways that promote student thinking and engagement."

Chase Orton
Author, *The Imperfect and Unfinished Math Teacher*
Independence, OR

"This book has so many great curricular and non-curricular tasks! I love how the authors answer frequently asked questions in the book based on their experience and research. It makes me feel like a thinking classroom *IS* possible in my classroom."

Howie Hua
Mathematics Instructor, Fresno State
Fresno, CA

"This masterpiece serves as an invaluable resource for grades 6–12 mathematics educators everywhere. Its updated research and rich tasks will empower both beginning and seasoned teachers with making a thinking classroom an immediate reality."

Tim Brzezinski
Middle and High School Mathematics Teacher,
Independent Mathematics Education Consultant
New Haven, CT

"*Mathematics Tasks for the Thinking Classroom: Grades 6–12* by Liljedahl and Webb is an indispensable resource. It offers fresh insights and practical ideas, seamlessly building on Building Thinking Classrooms. With 50 detailed tasks, the book highlights the importance of details in fostering deep, collaborative mathematical thinking. A must-have for educators committed to truly engaging students."

Alex Overwijck
Mathematics Teacher
Ottawa, Ontario, Canada

"Liljedahl and Webb have crafted a resource packed both with detailed tasks and clear explanations of Building Thinking Classroom practices that tackle teachers' common questions and point to subtle and significant nuances to enrich all students' learning experiences. This book is an essential tool to return to again and again."

Aleda Klassen
Secondary Mathematics and English Teacher
Waterloo, Ontario, Canada

"Peter Liljedahl and Kyle Webb do an amazing job providing updates to the original practices and concrete examples teachers have been asking for. The full lessons are an invaluable resource for teachers as we try tasks and tweak or build our own. This book is a fantastic resource for any educator on their BTC journey!"

Abigail Bates
Mathematics Teacher, Bear Creek High School
Stockton, CA

Mathematics Tasks for the Thinking Classroom

Grades 6–12

For every teacher who had the courage to change.

Mathematics Tasks for the Thinking Classroom

Grades 6–12

Peter Liljedahl

Kyle Webb

Illustrations by
Laura Wheeler

CORWIN

FOR INFORMATION:

Corwin
A Sage Company
2455 Teller Road
Thousand Oaks, California 91320
(800) 233-9936
www.corwin.com

Sage Publications Ltd.
1 Oliver's Yard
55 City Road
London EC1Y 1SP
United Kingdom

SAGE Publications India Pvt. Ltd.
Unit No 323-333, Third Floor, F-Block
International Trade Tower Nehru Place
New Delhi 110 019
India

SAGE Publications Asia-Pacific Pte. Ltd.
18 Cross Street #10-10/11/12
China Square Central
Singapore 048423

Vice President and Editorial Director: Monica Eckman
Associate Director and Publisher, STEM: Erin Null
Senior Editorial Assistant: Nyle De Leon
Production Editor: Tori Mirsadjadi
Typesetter: C&M Digitals (P) Ltd.
Proofreader: Wendy Jo Dymond
Indexer: Integra
Cover Designer: Scott Van Atta
Marketing Manager: Margaret O'Connor

Printed in the United States of America

Paperback ISBN 978-1-0718-7721-0

This book is printed on acid-free paper.

25 26 27 28 29 10 9 8 7 6 5 4 3 2 1

CONTENTS

Acknowledgments xv

About the Authors xvii

INTRODUCTION 1

WHO IS THIS BOOK FOR? 5

STRUCTURE OF THE BOOK 7

HOW TO READ THIS BOOK 11

PART 1: IN REVIEW AND IN THE NEW 14

CHAPTER 1: WHAT KINDS OF TASKS TO USE 17

HALLMARKS OF A GOOD THINKING TASK 17

MANIPULATIVES AS SUPPORT FOR THINKING TASKS 20

THE USE OF TECHNOLOGY WITH THINKING TASKS 25

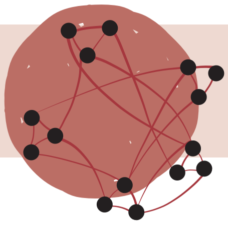
FAQ 26

QUESTIONS TO THINK ABOUT 29

CHAPTER 2: HOW, WHEN, AND WHERE TO GIVE TASKS 30

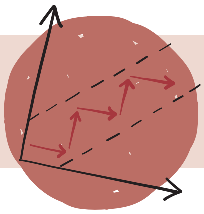
HOW TO GIVE THINKING TASKS 30
WHEN TO GIVE A THINKING TASK 32
WHERE TO GIVE A THINKING TASK 33
FAQ 34
QUESTIONS TO THINK ABOUT 36

CHAPTER 3: HOW TO BUILD AND UTILIZE STUDENT AUTONOMY 37



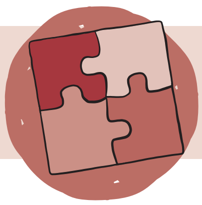
A NEW WAY OF DEVELOPING STUDENT AUTONOMY 37
FAQ 39
QUESTIONS TO THINK ABOUT 42

CHAPTER 4: HOW TO USE HINTS AND EXTENSIONS TO MAINTAIN FLOW 43



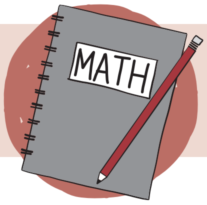
HOW ARE THINKING AND ENGAGEMENT CONNECTED? 43
MAINTAINING FLOW THROUGH TIMING AND SEQUENCE TYPES 45
FAQ 51
QUESTIONS TO THINK ABOUT 54

CHAPTER 5: HOW TO CONSOLIDATE THE TASK(S) 55



THREE KINDS OF CONSOLIDATION 56
CONSOLIDATION THROUGH A GUIDED GALLERY WALK 58
CONSOLIDATION THROUGH TEACHER-SCRIBE 60
FAQ 62
QUESTIONS TO THINK ABOUT 65

CHAPTER 6: HOW TO HAVE STUDENTS MAKE MEANINGFUL NOTES 66



COMMON BELIEFS ABOUT THE VALUE OF NOTES 66

NEW RESEARCH ON STUDENT NOTE MAKING 67

FAQ 71

QUESTIONS TO THINK ABOUT 72

CHAPTER 7: HOW TO HAVE STUDENTS DO CHECK-YOUR-UNDERSTANDING QUESTIONS 73

WHY HOMEWORK DOESN'T WORK 73

RESEARCH-BASED IMPROVEMENTS ON CHECK YOUR UNDERSTANDING QUESTIONS 75

FAQ 78

QUESTIONS TO THINK ABOUT 80

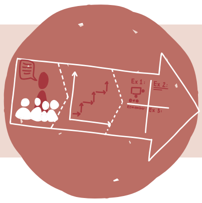
CHAPTER 8: HOW TO USE TASKS AS THE CONTEXT TO IMPROVE STUDENT COMPETENCIES 81

HOW DO WE REINFORCE THE COMPETENCIES WE WANT TO SEE IN A THINKING CLASSROOM? 81

FAQ 88

QUESTIONS TO THINK ABOUT 90

CHAPTER 9: WHAT DOES A LESSON LOOK LIKE IN A THINKING CLASSROOM? 92



THE THREE-PHASE LESSON FORMAT 94

FAQ 97

QUESTIONS TO THINK ABOUT 98

PART 2: NON-CURRICULAR TASKS 100

TASK #1. HOW MANY TIMES? 117
TASK #2. BILLIARD BOUNCE 124
TASK #3. 1001 PENNIES 134
TASK #4. FOLLOW THE LINE 140
TASK #5. TRIANGLE REVERSAL 148
TASK #6. SPLIT 25 155
TASK #7. MARBLE BAG 163
TASK #8. THE PIRATES AND THE DIAMOND 169
TASK #9. PAINTED CUBE 178
TASK #10. SUMTHINGS ARE IMPOSSIBLE 186
TASK # 11. SQUARING IS CARING 193
TASK #12. 3D TIC-TAC-TOE 201
TASK #13. AT ODDS WITH THE EVENS 210
TASK #14. KAYAK 218
TASK #15. DOUBLE-UP 227
TASK #16. PLAY DATE 236
TASK #17. PARTY CRASHER 243
TASK #18. BROKEN GLASS 251
TASK #19. FAN FUN 258
TASK #20. THIS LINE IS SO LONG 266

PART 3: CURRICULAR TASKS 272

TASK #21. COORDINATE PLANE 293
TASK #22. LCM AND GCF 298
TASK #23. INTRODUCTION TO PERCENTAGES I 303
TASK #24. INTRODUCTION TO PERCENTAGES II 308
TASK #25. SCALE DRAWINGS I 312

TASK #26. SCALE DRAWINGS II 320

TASK #27. AREA OF COMPOUND SHAPES 325

TASK #28. PYTHAGOREAN FORMULA 332

TASK #29. SOLVING MULTI-STEP EQUATIONS 341

TASK #30. LINEAR RELATIONS 346

TASK #31. FUNCTION STORIES 351

TASK #32. LINEAR EQUATIONS 362

TASK #33. SYSTEMS OF LINEAR EQUATIONS 369

TASK #34. SOLVING INEQUALITIES 377

TASK #35. GRAPHING LINEAR INEQUALITIES 385

TASK #36. GRAPHING SYSTEMS OF LINEAR INEQUALITIES 395

TASK #37. FACTORING QUADRATICS 403

TASK #38. SOLVING AND GRAPHING QUADRATICS 408

TASK #39. COMPLETING THE SQUARE 412

TASK #40. WEIGHTED AVERAGE 417

TASK #41. ARITHMETIC SERIES 422

TASK #42. SIMPLIFYING RADICALS 427

TASK #43. SOLVING IRRATIONAL EQUATIONS 431

TASK #44. SECTORS OF A CIRCLE 437

TASK #45. SOLVING WITH GEOMETRIC PROBLEMS 445

TASK #46. JUSTIFYING WITH GEOMETRIC PROBLEMS 452

TASK #47. TANGENT RATIO 461

TASK #48. TRIGONOMETRIC RATIOS 470

TASK #49. DIFFERENTIATING TRIGONOMETRIC FUNCTIONS 479

TASK #50. IMPLICIT DIFFERENTIATION 483

PART 4: FROM PAGE TO PRACTICE 490

WHERE TO FIND MORE TASKS 491

HOW TO THIN-SLICE FROM SCRATCH 497

HOW TO THIN-SLICE EXISTING CURRICULUM RESOURCES 510

HOW TO THIN-SLICE WORD PROBLEMS 518

HOW TO THICK-SLICE 523

Epilogue 528

References 530

Index 531

online resources

Visit the companion website at
https://companion.corwin.com/courses/BTC612Tasks
for downloadable resources.

ACKNOWLEDGMENTS

ACKNOWLEDGMENTS FROM PETER

I want to thank my wife, Theresa, for her constant companionship as we, together, bring *Building Thinking Classrooms* to teachers around the world. Without your support, none of this would have been possible. I want to thank Erin Null, my editor for all things BTC related. I value your support, your advocacy, your advice, your patience, but most of all your friendship. I also want to thank my co-author, Kyle. Your collegiality and friendship has helped to make this a better book. Finally, I want to thank all the teachers, consultants, and administrators who continue to explore with, trust in, and advocate for Building Thinking Classrooms. You are making a difference in the lives of students.

ACKNOWLEDGMENTS FROM KYLE

First and foremost, I want to thank my wife, Caroline. Your unwavering support, your enthusiasm for all my endeavors, and your absolute confidence in me are the foundation that allows me to do the work I love. Thank you. To my sons, Finn and Felix: watching you learn and grow inspires everything I do, and I hope the ideas in this book help shape your own learning journeys. I also want to thank Peter for inviting me on this incredible journey and for constantly pushing my thinking and inspiring me to be a better educator. Your mentorship, friendship, and faith in me mean more than you'll ever know. A special thanks to Maegan Giroux for her support and guidance and to the entire BTC team for their encouragement. My thanks also go to RCSD and MICDS for providing the professional opportunities that have shaped my career. Finally, thank you to every teacher who has trusted me to be a part of their learning journey; I hope I've given back a fraction of the impact you've all had on me.

ACKNOWLEDGMENTS FROM PETER AND KYLE

We also want to acknowledge the entire team at Corwin Press beginning with (again) our editor and Corwin Mathematics Publisher, Erin Null. Your ongoing support and advocacy for all things Building Thinking Classroom is what keeps the work moving forward. We want to thank Laura Wheeler for illustrating this book. You have, again, found ways to give life to our words through your artistry. And we thank Senior Editorial Assistant Nyle De Leon and Senior Project Editor Tori Mirsadjadi, who helped with many technical aspects of turning a manuscript into an actual book. Finally, we are grateful to Senior Marketing Manager Margaret O'Connor who worked so hard to get this book into your hands.

ABOUT THE AUTHORS

Dr. Peter Liljedahl (he/him) is a professor of mathematics education in the Faculty of Education at Simon Fraser University, author of the best-selling book, *Building Thinking Classrooms in Mathematics (Grades K–12): 14 Teaching Practices for Enhancing Learning*, and co-author of the best-selling book, *Mathematics Tasks for the Thinking Classroom (Grades K–5)*. Peter is a former high school mathematics teacher who has kept his research interests and activities close to the classroom. With a passion for fostering deep mathematical thinking and problem-solving skills, Peter has dedicated his career to reshaping classroom environments. His work emphasizes thinking, collaborative learning, and problem solving, leading to more effective and meaningful mathematics education experiences for students of all ages. Through his research-based innovative teaching methods, Peter continues to inspire educators worldwide to create dynamic and thought-provoking learning spaces. He consults regularly with teachers, schools, school districts, and ministries and departments of education on issues of teaching and learning, problem solving, assessment, numeracy, and building thinking classrooms. Peter has authored or co-authored 15 books, 45 book chapters, and 49 research articles on a wide range of topics including creativity in mathematics, the role of beliefs in the teaching and learning mathematics, and building thinking classrooms. He is the recipient of the Cmolik Prize for the Enhancement of Public Education in Canada (2017), the Margaret Sinclair Memorial Award Recognizing Innovation and Excellence in Mathematics Education in Canada (2018), and the Læringsprisen for Changing the way we think about Education in Denmark (2022).

Kyle Webb (he/him) is a teacher, learning consultant, and professional development leader based in White City, Saskatchewan, Canada. Never satisfied with the status quo, Kyle's passion is improving learning experiences for all students. With experience teaching grades 6 through 12, and in his roles as a teacher coach and school division numeracy consultant, Kyle works to transform classrooms. He has spearheaded the successful implementation of Building Thinking Classrooms (BTC) and modern assessment practices, and he has directly supported hundreds of teachers in implementing BTC through modeling, co-teaching, and collaborative planning. Working alongside teachers, he empowers them to refine their pedagogical approaches and foster environments where all students can thrive. Beyond direct classroom support, Kyle regularly leads Building Thinking Classrooms workshops, professional development sessions, and hosts and produces the *Think Thank Thunk* podcast, allowing him to share insights on teaching and extend the global reach of BTC practices. Kyle's commitment to improving teaching and learning within classrooms continues to shape the future of learning in Saskatchewan, across Canada, and beyond.

******

For information about consulting on Building Thinking Classrooms, go to https://www.buildingthinkingclassrooms.com/consulting

INTRODUCTION

If you have picked up this book, you are likely already familiar with the pedagogical framework known as Building Thinking Classrooms (BTC; Liljedahl, 2021). You probably also know that BTC is the result of more than 15 years of research grounded in the classroom practices that either support or hinder student thinking. The original research was a reaction to a recognized reality that the majority of students spend the majority of their time in a math classroom not thinking. And this is a problem. Thinking is a necessary precursor to learning and if students are not thinking, they are not learning. Something is broken and many who work in math education have been trying to repair it for more than half a century.

Since the introduction of the National Council of Teachers of Mathematics Principles and Standards (2000), math teachers and leaders, organizations, school districts, and departments and ministries of education around the world have been trying to change this reality. But, for the most part, they have been trying to make this change while not making changes to the classroom environment. Teaching through problem solving, problem-based learning, project-based learning, and inquiry initiatives coupled with changes to assessment, the introduction of competencies, mathematical processes, and mathematical mindset work have dominated curriculum reform for the last 50 years. But students are still sitting, and teachers are still standing. Students are still writing on paper, while teachers are still writing on whiteboards. And as much as collaboration has permeated education, learning has largely been seen as a solitary activity.

The work on BTC changed all this. It began by looking at the core routines that every teacher does:

- the types of tasks we use;
- how we form collaborative groups;
- where students work;
- how we arrange the furniture;
- how we answer questions;
- when, where, and how tasks are given;
- what homework looks like;
- how we foster student autonomy;
- how we use hints and extensions;
- how we consolidate a lesson;

- how students take notes;
- what we choose to evaluate;
- how we use formative assessment; and
- how we grade.

These routines account for 90% of what every teacher does (Kaplinsky, 2022). My question was, can these routines, individually and collectively, be enacted differently so as to maximize student thinking? And thus began a journey that, so far, has taken more than 15 years—a journey in pursuit of optimal ways to enact each of these core routines in such a way as to maximize student thinking.

The results of this research were transformative. Classrooms went from spaces where students sat working in their individual notebooks to spaces where they stood in random groups of three working on vertical whiteboards. They went from spaces where we, as teachers, pre-taught students how to do things like find the area of a parallelogram, calculate percentages, and factor quadratics to places where students figured these things out with the support of their random groups and the groups around them before being taught formally. Classrooms went from spaces where students were told meaning to spaces where they were making meaning. And they went from spaces where only 20% of students were thinking for 20% of the time to spaces where upward of 90% of students were thinking for 100% of the time. And the 10% who weren't thinking for 100% of the time were still thinking for more than 50% of the time.

What BTC did, that so many other initiatives had not, was examine the role that environment plays on student behavior. By examining the core institutional structures, BTC revealed that so many of the things about classrooms that have been invariant for more than a century have outlived their utility. Having student sit in their desks worked well in the post-industrial model of education when the goals of education were conformity and compliance, and school served as a vehicle for preparing factory workers. But if the goal is to think, we are going to have to do things differently. What BTC showed us was that to achieve 21st century goals we needed to get away from the 19th century classroom practices—we needed new practices. The research into how to build a thinking classroom told us that if we want our students to think, then we need to

- use thinking tasks;
- form frequent visibly random groupings;
- use vertical non-permanent surfaces;
- defront the classroom;
- only answer keep thinking questions;
- give tasks early, standing, and verbally;
- give check-your-understanding questions;

- be intentionally less helpful;
- create and manage flow;
- consolidate from the bottom;
- use meaningful notes;
- evaluate what you value;
- communicate to students where they are and where they are going; and
- report out based on data (not points).

Each of these empirically deduced practices is a response to the question of how we can enact our core classroom routines so as to maximize student thinking. The research into BTC began in 2003. By 2014, some of the optimal thinking practices had begun to emerge from the data, each of which was obvious and surprising, liberating and frightening, simple and complex. And this pattern continued. Each thinking practice that emerged from the data was forehead-slapping obvious (if we want students to think, then we have to give them something to think about) and complex (what constitutes a good thinking task and how do we find and/or make them)—far too complex to be captured in the list above, a keynote, or an article. This is not to say that efforts were not made. I gave talks and wrote articles—neither of which could fully encapsulate the complexity of each practice. The only ways teachers could really learn about the nuances was through workshops—access to which were hampered by vastness of geography and the limits of time.

In October 2020 the book *Building Thinking Classrooms in Mathematics: 14 Teaching Practices for Enhancing Learning* (Liljedahl, 2021) was released. For the first time, research that had taken, to that point, more than 15 years to collect, analyze, and make sense of now lived in a form that allowed teachers to access it without having to be in a workshop. And access it they did. Since publication, BTC has taken on a life of its own. Teachers have begun building their own thinking classrooms and students have begun experiencing math as something to be thought about—in groups and on their own. Students are more often now standing rather than sitting, writing on whiteboards rather than paper, working in groups rather than individually, and even working in groups of groups rather than groups working in isolation (see Figures i.1 and i.2).

Figure i.1 | Eleventh grade students working together in groups of three at whiteboards.

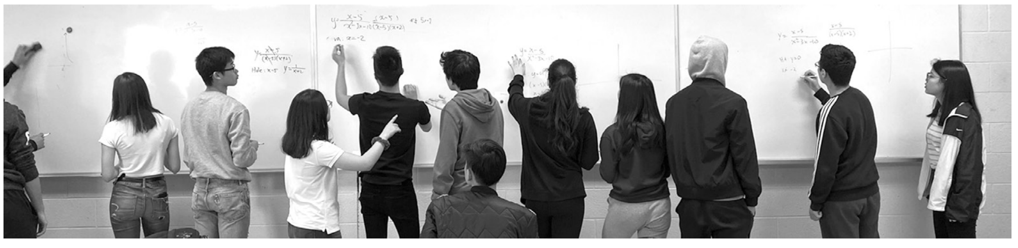


Photo credit: Peter Liljedahl

Figure i.2 | Students in a sixth/seventh-grade combination class working together in groups of three at whiteboards.

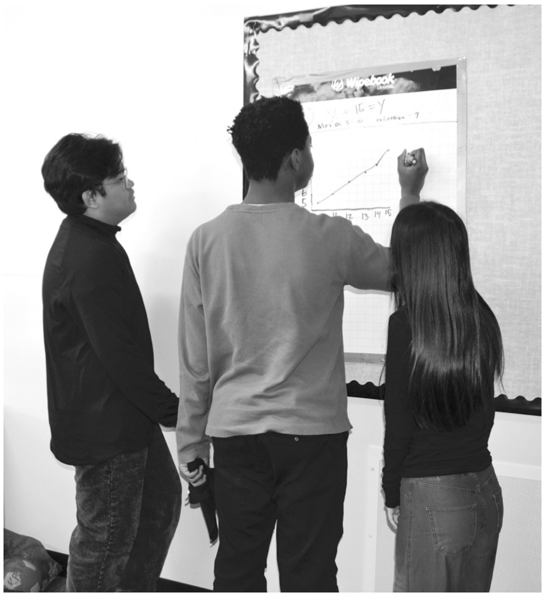


Photo credit: Erin Null

More classrooms have been defronted, tasks are being given verbally to groups of students huddled around the teacher, and flow is being maintained through hints and extensions. And so much more. The book—and the book clubs, Facebook groups, and PLCs that have sprung up around BTC—enabled teachers all over the world to build their own thinking classroom without having to attend a workshop with me. This is not to say that he became obsolete. Countless requests to visit book clubs, to lead workshops, and to give talks started pouring in. Teachers had questions. They assumed I had all the answers.

At the same time, school districts, divisions, and boards the world over began systematically supporting teachers in their pursuit to implement thinking classrooms. Some did this on their own, using only the book and publicly available podcasts, interviews, and presentations I had given over the years. Others asked me to provide workshops to their teachers—including a collection of school divisions in and around Regina, Canada, whom I partnered with starting in the fall of 2021 to provide ongoing workshops to groups of teachers wanting to implement the BTC practices. This is where I met Kyle Webb.

Kyle was the Numeracy Learning Consultant for the Regina Catholic School Division and worked hard to support teachers in their implementation of BTC in their K–12 classrooms. In this work, Kyle spent years walking alongside teachers as they implemented the 14 BTC practices. He taught and cotaught countless lessons and gave numerous workshops within his school division and beyond.

Across these many interactions, one of the questions we encountered the most was, *"Where can we get more tasks?"* The *Building Thinking Classrooms in Mathematics* book had some, and it referenced where teachers could find additional tasks. But teachers wanted more. And until now, our response to this has been *"No you don't."* We are drowning in tasks. Our textbooks are full of them and tasks are posted on twitter and Facebook in droves. Kyle, together with Maegan Giroux, had complied a further 800+ tasks (https://tasks.kylewebb.ca/). In fact, if we do a Google search for "*problem of the day*" we get 7 trillion hits. We do not need more tasks. And yet the requests kept coming.

After some time, we began to realize that the request for more tasks was not to be taken literally—it was a proxy. Teachers weren't asking where they could get more tasks. They were asking

1. where they could find more good tasks—tasks that had been tested in classrooms, with real students, and had been shown to work;
2. where they could find tasks that could help them build their own thinking classroom;
3. where they could find tasks that address specific curriculum outcomes or standards; and
4. where they could learn how to implement specific tasks to ensure they were successful.

That's what this book is about—tasks. Lots of tasks—tasks that have been tested in diverse classrooms with real students and have been shown to be effective at shifting student engagement, transforming student thinking and learning behavior, and promoting learning of specific curricular outcomes and standards. Lots of tasks—but not so many that you have difficulty deciding which ones to choose.

Here's the thing, though—tasks are inert and incapable of achieving any of these goals on their own. They need pedagogy to bring them to life. So, although this book is ostensibly about tasks, what this book is *really* about is how to use these tasks to achieve the pedagogical goals of BTC and the curricular goals set by your jurisdiction. As such, this book also offers a deep dive into several of the BTC practices and discusses how to use these practices to breathe life into each of the tasks in this book. At the same time, these tasks also become the context through which we can work on enhancing our familiarity and competency with these BTC practices.

What this book is *really* about is how to use these tasks to achieve the pedagogical goals of BTC and the curricular goals set by your jurisdiction.

WHO IS THIS BOOK FOR?

This book has been written for five distinct, but related, audiences. First, this book is for the teacher who has read *Building Thinking Classrooms in Mathematics: 14 Teaching Practices for Enhancing Learning*, has used the macro- and micro-moves detailed in each chapter to build their own thinking classroom and is now looking for more tasks to feed their thinking classroom. Second, this book is for the teacher who has read the BTC book, has not yet begun to implement the BTC practices in their classroom, but is looking for more tasks in anticipation of doing so.

If you have read the main BTC book, you are aware that it contains very good tasks for building and maintaining your thinking classroom. And if you have been following the BTC movement through social media you may also be aware that, although the tasks in the main BTC book are good, these are now tasks that are *"out there,"* and many students have already seen them. This was unlikely when the BTC book was first released in 2020, but as BTC has become more and more popular, there is greater likelihood that your students have previously been in a thinking classroom in some shape or form, and there is greater likelihood that they have been exposed to the tasks in the BTC book. Maybe last year's teacher used them in an effort to build their thinking classroom. Or maybe, you were last year's teacher, and you have, for example, shifted from teaching grade 7 to grade 8 or from teaching in Algebra II class to a precalculus AP class. Regardless, tasks like *The Tax Collector* (BTC, p. 107), *Gold Chain* (BTC, p. 81), and *Country Road* (BTC, p. 228) have become very popular among grades 6 through 12 teachers and are being used in classrooms all over the world—whether in a thinking classroom or not. More good tasks are needed for you to build and sustain your thinking classroom. If this is what you are looking for, then this book is for you.

The third audience, like the first two, is teachers who have read the BTC book and used the macro- and micro-moves detailed in each chapter to build their own thinking classroom. Unlike the first two audiences, however, they are looking not for more tasks—although more tasks are nice—but for more ideas on how to implement the BTC practices, including how to implement curricular tasks, more examples of thin-slicing, more examples of consolidation, more ideas for check-your-understanding (CYU) questions, and more in-depth information on implementing meaningful notes.

Although the BTC book was published in 2020, the ideas that are in it were, so to say, locked in in 2019. Since that time, the research on BTC has not stood still. I have continued to work in classrooms with real teachers and real students. And new practices have emerged. These practices are mostly in the form of micro-moves—important moves that are proving to make the macro-moves easier to implement and more effective in engaging students in thinking. Some of these new practices are in this book. If this is what you are looking for, then this book is also for you.

The fourth audience is the teachers who may know little to nothing about thinking classrooms, have not read the BTC book, but who have been working on crafting a student-centered, problem-based classroom and are in search of good tasks. If this is what you are looking for, then this book is also for you. However, you will find more than what you were looking for as each of the tasks in this book is wrapped in BTC practices that may help you to be more successful in achieving your goals and may move you to want to learn more about *Building Thinking Classrooms*.

Finally, this book is also for the "middle children" (Vardabasso, 2023) of education—the consultants, coaches, coordinators, numeracy leads, and so on. It is for those who are not teachers but work with teachers, and, as such, are in need of tasks they can use to introduce and showcase the best and most authentic picture of a thinking classroom.

STRUCTURE OF THE BOOK

To be able to address each of the five aforementioned audiences, this book contains four parts:

- Part 1: BTC Practices: In Review and in the New
- Part 2: Non-Curricular Tasks
- Part 3: Curricular Tasks
- Part 4: From Page to Practice

Part 1: BTC Practices—In Review and in the New

Part 1 of the book provides a brief review of the eight BTC practices that are most relevant to the use of tasks:

1. What Kinds of Tasks to Use (BTC, Chapter 1),
2. How, When, and Where to Give Tasks (BTC, Chapter 6),
3. How to Build and Utilize Student Autonomy (BTC, Chapter 8),
4. How to Use Hints and Extensions to Maintain Flow (BTC, Chapter 9),
5. How to Consolidate the Task(s) (BTC, Chapter 10),
6. How to Have Students Make Meaningful Notes (BTC, Chapter 11),
7. How to Have Students do Check-Your-Understanding Questions (BTC, Chapter 7), and
8. How to Use Tasks as The Context to Improve Student Competencies (BTC, Chapter 12).

Each chapter culminates with answers to some frequently asked questions (FAQ's). Part 1 of the book ends with a chapter on how to pull all these practices together into a lesson—What Does a Lesson Look Like in a Thinking Classroom (BTC, Chapter 15).

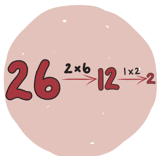
Part 1 is both more and less than what the corresponding chapters in the BTC book offer. More, because it offers new research results leading to new nuances and insights in some of these practices. Less, because, new insights aside, it is still just a review. To get a full depth of understanding of each of these practices you really need to have read the corresponding chapters in the primary BTC book. Regardless, what Part 1 does is set you up to best be able to extract the full affordances that each of the tasks in Parts 2 and 3 of the book have to offer.

Part 2: Non-Curricular Thinking Tasks

Part 2 is a collection of 20 non-curricular tasks, each of which goes *way* beyond just the task. Each task begins with a statement of the task in its simplest form followed by five specific indicators that will help you quickly select which non-curricular tasks are right for you and your students at this specific point in your journey to build a thinking classroom:

- Grades—what grade level(s) is the task appropriate for
- Content Potential—what mathematics topics may students encounter while solving this task
- Perseverance Scales—how much perseverance will the students need to solve this task for a variety of suitable grades
- Macro-moves—which of the BTC 14 practices are specifically supported by the task
- Competencies—what student competencies can be introduced and/or enhanced by engaging in this task

After these indicators comes the actual task presented in a comprehensive format that will help you launch, facilitate, and consolidate the task.



TASK 1: HOW MANY TIMES?

TASK

Take a 2-digit number. Multiply its digits together. If the result is a 2-digit number, repeat this process until you get a one-digit number. Which number will allow this process to go on the most times?

Grades: 6–8

Content Potential: multiplication

Perseverance Scale:

Grade Level	6	7	8
Perseverance Level	1	1	1

Macro-moves: answer only keep thinking questions, mobilizing knowledge

Competencies: organization, pattern spotting

Source: Adapted from Sloane (1973).

This includes:

- Launch Script—a detailed script on how to launch the task with your students
- Creating Access—a set of notes about how to create access to the task for different grade or ability levels

- Extension Script—a detailed script on how to extend groups when they have solved the original, and subsequent, problems
- Author Solution(s)—a discussion of the ideas central to the solution of the task
- Student Solutions—a set of possible student solutions.
- Suggested Hints—also accompanying the solutions is a set of hints that can be used to help move students from one solution to another
- Consolidation—a discussion of what is critical to pull out of the solutions during consolidation as well as, embedded within the sequence of student solutions, the order in which the task can be consolidated with key solution elements highlighted
- Check-Your-Understanding Questions—new types of CYU questions (discussed in Part 1 of the book) that can be used to help students assess and consolidate their individual understanding from the task
- Author Notes—helpful comments, suggestions, tips, and tricks about the launch, facilitation, and consolidation of the task gleaned from multiple experiences using the task with students
- Notes to My Future Forgetful Self—a space for you to add your own helpful comments, suggestions, tips, and tricks about the launch, facilitation, and consolidation of the task gleaned from your own experiences using the task with your students

Taken together, these tasks provide you with the comprehensive information you need regardless of why you are reading this book. Whether you are just looking for good tasks, more tasks for your BTC classroom, or wanting to refine and enhance your BTC teaching practices, the detailed presentation of each task will give you what you need. And if you want more tasks, Part 4 of the book provides a catalogue listing resources for more tasks along with access to a task template for you to explore and build out your own indicators for use.

Part 3: Curricular Thinking Tasks

Part 3 is a collection of 30 curricular tasks. Each of these is built out from an initial question into a sequence of questions that get progressively more challenging as students abilities increase (called thin-slicing). Unlike non-curricular tasks, however, by the time you get to these curricular tasks, there is an assumption that most of the BTC practices are well established within your classroom norms and routines. As such, the focus on curricular tasks is less on how we teach (BTC practices) and more on what we teach (content and competencies). This is not to say that you should stop working on the BTC practices. These take months for both you and your students to become proficient and comfortable with. But in this part of the book, the focus is more on the content.

To this end, in order to help you decide whether the content of a particular task is appropriate for you and your students, each task begins with a brief description of what the task sequence is. This is followed by four salient indicators to help you locate the task sequence within your curriculum and your progress in a thinking classroom:

- Content—the mathematics topics the task covers
- Competencies—which student competencies can be introduced and/or enhanced by engaging in this task sequence
- Seen Before—a list of topics that students have *previously seen*
- Before You Launch—tips to be aware of before you launch

After the content and competency indicators comes the actual task sequence presented in a format that will help you launch, facilitate, and consolidate the sequence:

- Launch Script—a detailed script on how to launch the task with your students
- Task Sequence—a list, broken into types, of progressively more challenging questions (thin-sliced questions) that the students can move through within one lesson or concept
- Hints—occasional hints that can be used to help move students' thinking as you move them from one type of question to another
- Consolidation Tasks—a set of three tasks that can be used to facilitate a new type of consolidation (discussed in Part 1 of the book) specific to these types of (thin-sliced) curricular tasks
- Student Notes to Their Future Forgetful Self—a new template (discussed in Part 1 of the book) for having students make meaningful notes as a way to organize their thinking from the activity as well as to create a record of this thinking
- Check-Your-Understanding Questions—the task concludes with a new format of related CYU questions (discussed in Part 1 of the book) that can be used to help students assess and consolidate their individual understanding from the task
- Author Notes—helpful comments, suggestions, tips, and tricks about the launch, facilitation, and consolidation of the task gleaned from multiple experiences using the task with students
- Notes to My Future Forgetful Self—a space for you to add your own helpful comments, suggestions, tips, and tricks about the launch, facilitation, and consolidation of the task gleaned from their own experiences using the task with their students

Although each of these curricular tasks can be used to take students through specific topics across a wide range of grades and curricula, this list is not comprehensive. That is, there is no grade or curricula that is entirely covered by these 30 examples. This means that you are going to have to start making your own (thin-sliced) sequences—either from scratch or from existing resources—of tasks to use with students on topics that you teach. Part 4 of the book will provide you with help to do this.

Part 4: From Page to Practice

Part 4 is going to help you move beyond the book and shows you where else you can find good tasks—both curricular and non-curricular. More importantly, Part 4 will teach you how to make your own thin-sliced sequences of curricular tasks. This part of the book is broken into five sections:

- Where to Find More Tasks
- How to Thin-Slice from Scratch
- How to Thin-Slice Existing Curriculum Resources
- How to Thin-Slice Word Problems
- How to Thick-Slice

Each of these sections is about finding and/or making tasks for the thinking classroom—both curricular and non-curricular. Each section ends by providing you with task templates into which you can embed your work to build a living archive of tasks for your—and other's—thinking classroom(s).

HOW TO READ THIS BOOK

Who you are will determine, to a great extent, how you should read this book. Regardless, we urge all of you to read Part 1. Whether you are new to BTC or a seasoned user, there are new things in Part 1 that you need to understand to make sense of Parts 2 and 3.

If you are just beginning, go to Part 2, which contains all the non-curricular tasks you need to build a culture of thinking in your classroom. The indicators at the beginning of each task will help you decide which tasks are appropriate for your students. Keep in mind that these indicators are only a guide and you need to reflect realistically about where you and your students are in your journey to build a thinking classroom. Also keep in mind that your class is not a homogeneous collection of students all in possession of the same interests, curiosities, and prerequisite abilities. When in doubt, start easier and allow the class as a whole to show you what they are capable of and ready for. Once you have done four to six non-curricular tasks with your students move onto Part 3.

If you already have a culture of thinking or have just recently developed one, then Part 2 still has value to you—just not right now. But it will be useful at the start of a new school year when you need to build up a culture of thinking anew, or when your already thinking students come back from an extended break and they need to be re-immersed into a culture of thinking. For the time being, however, you may want to jump to Part 3.

As in Part 2, the indicators in Part 3 will help you to find appropriate thin-sliced sequences of questions for your students. There are 30 tasks spread across seven grades, which means that, although you will for sure find curricular thin-sliced tasks you can use in your lesson you will not find nearly enough to sustain you for an entire school year, but that is the point. These are designed to be examples to give you models for then making your own. This is what Part 4 of the book is for. But before you jump to Part 4, we urge you to examine each of the 30 curricular thinking task sequences. There are patterns in these tasks that will help make Part 4 make sense. In fact, we are confident that if you do examine each of the 30 curricular tasks and the patterns within the structures of the task you will be able to build your own curricular task sequences without too much help from Part 4.

Once you have looked at all of the curricular task sequences, go to Part 4 for resources for additional non-curricular tasks and helpful tips on how to build your own curricular task sequences. Here you will come to see that thin-slicing is really is not as difficult as you might have imagined, and you will quickly realize that you have the knowledge and the ability to sustain your thinking classroom through a whole year of curriculum.

In the meantime, enjoy the journey to building, sustaining, and leveraging your own thinking classroom. It will re-invigorate you as a teacher and you will find joy in watching students think, make meaning, and learn.

Peter Liljedahl & Kyle Webb

Notes to My Future Forgetful Self

PART 1

BTC PRACTICES: IN REVIEW AND IN THE NEW

I have been a math educator for about twenty years and *Building Thinking Classrooms in Mathematics* by Peter Liljedahl has more potential to improve the way we teach mathematics than any other book I have ever read. These are not words I say lightly. This book is an absolute game-changer for all math educators and everyone needs to read it.

—Robert Kaplinsky
Author of *Open Middle Math: Problems That Unlock Student thinking, Grades 6–12*

When the main *Building Thinking Classrooms in Mathematics* book (hereafter referred to as the BTC book or the main BTC book) was published, it contained the results of 15 years of research with over 400 teachers and the results of thousands of micro-experiments around what works in real classrooms to get students to think. But this was not the culmination. The research continued. In this part of the book, we summarize some of the results of this new research as well as the original research to bring you, the reader, up to speed on the practices that are relevant to Parts 2 and 3 of this book.

As mentioned in the introduction, this summary is both *less than* and *more than* the details in the main BTC book. It is *less than* in that this is, to some degree, a summary. As such, we do not go into all 14 of the BTC practices and for the eight that we do go into, we do not go into the same level of detail that the main BTC book does. We do not visit or revisit the FAQs that are in the main BTC book. So, as you read Part 1 of this book and you find yourself with questions or you find yourself saying, "Yeah, but . . . ," we strongly urge you to read, or reread, the relevant chapters in the original BTC book.

At the same time, this summary is *more than* the original book in that it presents exciting new results around consolidation (BTC, Chapter 10), meaningful notes (BTC, Chapter 11), and check-your-understanding questions (BTC, Chapter 7). Some of these results are relevant to both non-curricular tasks (Part 2 of this book) and curricular tasks (Part 3 of this book). Others are relevant to only curricular tasks. But these new results are built on the same foundational realizations that are presented in the original BTC book and are subject to the same FAQs. So, as you read about these new results, if you find yourself with questions or saying, "Yeah, but . . . ," we again urge you to read, or reread, the relevant chapters in the original BTC book.

With that in mind, in what follows is a summary of the eight BTC practices and one lesson framework that are most relevant to this book with its focus on tasks:

1. What kinds of tasks to use (BTC, Chapter 1),
2. How, when, and where to give tasks (BTC, Chapter 6),
3. How to build and utilize student autonomy (BTC, Chapter 8),
4. How to use hints and extensions to maintain flow (BTC, Chapter 9),
5. How to consolidate the task(s) (BTC, Chapter 10),
6. How to have students make meaningful notes (BTC, Chapter 11),
7. How to use check-your-understanding questions (BTC, Chapter 7),
8. How to use tasks as the context to improve student competencies (BTC, Chapter 12), and
9. What does a lesson look like in a thinking classroom (BTC, Chapter 15).

Missing from this list are summaries of the original BTC book's Chapter 2 (how we form collaborative groups—*frequent visibly random groups*) and Chapter 3 (where students work—*vertical non-permanent surface*). In this book we are taking as shared that we should be using visibly random groups and that students should be working at vertical non-permanent surfaces (VNPSs) every lesson. Nothing has changed with respect to these practices. They are cornerstones of BTC, and all other practices are built on the foundation that they anchor. If you are not familiar with these, we suggest you read, or reread, chapters 2 and 3 in the main BTC book.

Students don't listen to what we say. They listen to what we do.

The same is true for BTC Chapter 4 (how we arrange furniture—*we defront the classroom*) and Chapter 5 (how we answer question—*only answer keep thinking questions*). Like with random groups and vertical surfaces, these are practices that are vital to the building and sustaining of a thinking classroom and are not to be disregarded. Students don't listen to what we say. They listen to what we do. And holding onto the aesthetics of an overly organized classroom tells students that perfection is what matters. Thinking is not perfect. It is full of errors and wrong turns. And it is messy. A defronted classroom tells students that this is OK. Likewise, not answering students' proximity and stop thinking questions tells students that you have confidence in them—that they are all capable and all equal. Students listen to what we do. The principles of chapters 4 and 5 are not dispensable. If you have further questions about these, we suggest you read, or reread, chapters 4 and 5 in the main BTC book.

Also omitted are summaries of chapters 13 and 14 on formative and summative assessment of content, respectively. Like with the other chapters, these are important practices and are not dispensable. But, they are different. Unlike the rest of the practices in the main BTC book, formative and summative assessment have way more institutional constraints on them. How you assess can be dictated by outside forces. This is not true of how you form groups or how you answer questions. Further, assessment is a practice that occupies time at the scale of the unit. The other practices in the main BTC book occupy time at the scale of the lesson. This is not to say that assessment does not happen in the context of a lesson, but not every lesson. The focus of this book is on the lesson and as such, we have not woven assessment of content into it. Having said that, we still review and use rubrics for the assessment and development of student behaviors (BTC, Chapter 12). Unlike assessment of content, assessment of behaviors can happen in every lesson.

Having discussed what is not included in this book, what follows is a review of the eight practices and a lesson framework that are particularly relevant to the use of tasks—at the lesson level. Importantly, these reviews include some critical evolutions, updates, and nuances about these particular practices that are important to consider when getting students to think about mathematics using both non-curricular and curricular tasks. At the end of each short chapter, you will also find answers to the most Frequently-Asked-Questions (FAQs) we get from educators as well as some questions for you to think about.

CHAPTER 1

WHAT KINDS OF TASKS TO USE

If we want our students to think, we need to give them something to think about. And that something is a task. This seems obvious—maybe too obvious. But there is irony in this obviousness. There has been a huge movement in mathematics education for the last 30 years to make and find the ideal tasks. Much of this movement is predicated on the assumption that a good task will solve all our issues in math education—that what is missing are good tasks. It is very easy to get swept up in this idea. In fact, you may recall from the main BTC book, Peter's first encounter with Jane was around this very issue—good tasks. Jane wanted to introduce problem solving to her students and she assumed that all she needed was good tasks. Peter's response was to provide her with good tasks. The ensuing disastrous lessons proved that good tasks were not enough to get students to think—we need other things as well, and the main BTC book and the remainder of Part 1 of this book are all about what these other things are. But, we still need good tasks. While insufficient on its own, a good task is a necessary condition for building a thinking classroom. In what follows, we look closer at three elements of what helps make a task a good thinking task and what can enhance their use, including:

1. Hallmarks of a good thinking task
2. Manipulatives as support for thinking tasks
3. The use of technology with thinking tasks

HALLMARKS OF A GOOD THINKING TASK

So, what makes a good task? It turns out there are three distinct qualities that make a mathematical task a good thinking task:

1. it has a low floor,
2. it has a high ceiling, and
3. it is novel.

Again, this may sound obvious, but there are some important things to understand about these traits. First, a good task has a low floor (Papert, 1980)—a way for every student to be able to start. This is incredibly important. But why? Every jurisdiction that we have worked in in the last many years is striving to achieve equity within and through education. But what does this mean? How does this affect teachers' day-to-day practice? When we talk to teachers about this, they have a difficult time articulating what they are meant to be doing differently in their classroom and they often default to saying things like "making things equal," "inclusion," and "this is important." It is important.

Equity is what we want to achieve. Creating access is one of the ways we achieve it.

The reason they are having difficulty is because equity is a goal and equity is an outcome—but the day-to-day work of achieving the outcome of equity lies in creating access. Equity is what we want to achieve. Creating access is one of the ways we achieve it. This is something that is much easier for teachers to think about as they plan and execute their day-to-day teaching. "How can I create access for my students?" is a question teachers can answer. And this is why a good task needs to have a low floor. It creates access. To do otherwise would exclude some students in the classroom—it would make mathematics inaccessible. And inequitable.

Having a low-floor doesn't mean that the task should be trivial. The second trait of a good task is that it also has a high ceiling (Papert, 1980)—meaning that everyone will meet challenge at some point in the task. When and where this happens is dependent on the student, their past experience, and their prior knowledge, but everyone needs to meet some level of challenge at some point in the task. It is through challenge that thinking is really unlocked. It is in the moments when we are stuck that we think. Thinking is what we do when we don't know what to do. And by thinking through being stuck students will learn to persist, to fail and try again, and they will learn something about themselves and about mathematics.

Thinking is what we do when we don't know what to do.

There is a strong correlation between the thinking that occurs when we are stuck and the idea of *productive struggle* (SanGiovanni et al., 2020), which can be loosely defined as the willingness to persist through challenge and to keep trying even when progress is stalled. The work on Building Thinking Classrooms has shown that productive struggle is more of a state than it is a trait. Certainly, some students are more persistent than others, but even the most persistent student will give up under the wrong circumstances and even the most vulnerable student will persist under the right circumstances. But what are the right circumstances? When a student meets challenge on the heels of success, they are more likely to enter into a state of productive struggle—to persist, to endure, and to eventually overcome. If students meet challenge without prior success, or worse—on the heels of failure, they are more likely to give up. Early success is key to later perseverance and productive struggle.

Early success is key to later perseverance and productive struggle.

This further justifies the need for a good thinking task to have both a low floor and a high ceiling. The low floor not only ensures access; it also creates opportunities for early success. And the high ceiling creates the opportunity for productive struggle.

The third quality of a good thinking task is novelty—it has to be something that the students have not seen before. This is because thinking is what we do when we don't know what to do. If we know what to do, we just do it—we mimic. Novelty ensures that the students can't mimic. It also ensures that, at some level and in some way, they will get stuck. And that is when they will think. They will try and they will fail and then they will try again. And then they will get unstuck. And when they do, they will learn. They will learn about mathematics, they will learn about themselves, and they will learn how to think. Along the way, they will also learn that being stuck is not a bad thing, it is just part of mathematics.

Teacher: So, what's going on.

Student: We're stuck.

Teacher: So, what happens now?

Student: We try to get unstuck.

This is not to say that all of this—low floor, high ceiling, and novelty—needs to be achieved within one big meaty task. By and large, most good thinking tasks are actually composed of a series of smaller tasks that get progressively more challenging. This progression allows for both a low floor and a high ceiling and ensures that novelty can be achieved, even if the initial task may be familiar to the students. This will be discussed more in Chapter 4 and is the basis of all of the non-curricular and curricular thinking tasks in Part 2 and Part 3, respectively, of this book.

Because of the diversity of the classrooms in which we teach, even the richest tasks need extensions. Consider, for example, the following Open Middle task (Kaplinsky, 2019) intended for grade 7–10 students:

Using the digits 1 to 9 at most one time each, place a digit in each box to create an expression with the greatest possible value.

$$\frac{\square^{\square} \times \square^{\square}}{\square^{\square} \times \square^{\square}} =$$

This is a rich curricular task that has a low floor, high ceiling, and is likely novel to the students. Some students may start with trial and error—just plugging in digits to see if their new answer is more or less than their previous answer. Eventually, they may arrive at the conclusion that they need to make the numerator as large as possible and the denominator as small as possible. If they think about the denominator first, it is pretty clear that the smallest value can be arrived at if they use the digits 1–4 in some combination. But what is the correct combination? The students will eventually discover that the smallest denominator is 8 and is achieved from $1^4 \times 2^3 = 8$. They may also discover that there are lots of ways to get 8 by varying the exponent on the first term ($1^4 = 1^5 = 1^6 = \ldots = 1$).

One of the main tenants of a thinking classroom is that no one ever gets to be done.

Making the numerator as large as possible is not quite as simple. If students realize that to get the largest numerator, they will need to use the digits 6–9 in some combination, then they are off to a good start. But what is the biggest combination? Which is larger, $8^9 \times 7^6$ or $9^6 \times 8^7$? With the help of a calculator, they can find out. And along the way, they may discover there are only a total of 12 ways to combine 6–9 to make a numerator, each of which can be calculated and compared with the others. But, they may also discover that $8^9 > 9^8$ and that this turns out to be true for any combination of two digits from 6 to 9. That is, $6^9 > 9^6$ and so on.

After enough exploration and discovery, the students will discover that the largest numerator can be achieved from $6^8 \times 7^9$ and that the largest ration can be achieved from

$$\frac{6^8 \times 7^9}{1^4 \times 2^3}$$

One of the main tenants of a thinking classroom is that no one ever gets to be done. We will talk more about this throughout Part 1. In the meantime, for students who "finish" we give them an extension:

Using the digits 1 to 9 at most one time each, place a digit in each box to create an expression with a value as close to 1 as possible.

$$\frac{\square^{\square} \times \square^{\square}}{\square^{\square} \times \square^{\square}} =$$

It doesn't matter how rich the original task is. All tasks need extensions—something that requires the students to do more thinking, to persist longer, and to keep struggling. And they will. The success on the first tasks makes it more likely that they will enter into a state of productive struggle on the next one. And so on. As such, all the tasks in this book contain extensions.

MANIPULATIVES AS SUPPORT FOR THINKING TASKS

Another facet of good thinking tasks worth exploring are those that can be brought to life through manipulatives. Manipulatives are central to the teaching and learning of mathematics.

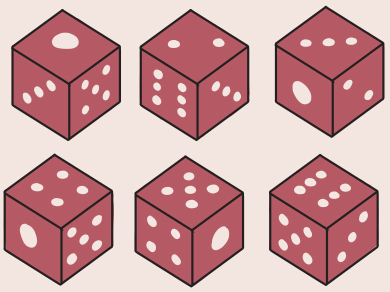
> *The evidence indicates, in short, that manipulatives can provide valuable support for student learning when teachers interact over time with the students to help them build links between the object, the symbol, and the mathematical idea both represent.*
>
> (National Research Council, 2001, p. 354)

This is true from kindergarten to grade 12. And this truth is not diminished in a thinking classroom where manipulatives should be just as ubiquitous as they are in any other math classroom. In fact, in a thinking classroom, the thinking tasks we use are sometimes inseparable from manipulatives. Whether we are using integer disks, fraction disks, fraction strips, algebra tiles, balance scales, mobiles, or 3D shapes, manipulatives support students' development of rational numbers, algebraic thinking, and shape. With respect to thinking tasks, manipulatives serve three distinct purposes:

1. Students think about the manipulatives.
2. Students think with the manipulatives.
3. Students represent their thinking with manipulatives.

Sometimes the manipulative is the context from which our tasks emerge. For example, consider dice (see Figure 1.1) and the following tasks:

Figure 1.1 | Dice

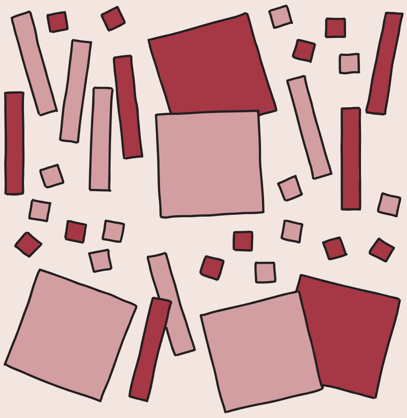


1. What patterns do you notice about the way numbers are placed on a dice?
2. Take 6 dice, shake them in your hand, and then stack them one on top of each other. Is there a quick way to find the sum of all the visible faces? Is there a quick way to find the sum of all the invisible faces?
3. Given what you noticed about how numbers are placed on a dice, are all dice the same? That is, how many different dice can you make while adhering to this rule about how numbers need to be placed on a dice?

These three tasks emerge out of the manipulatives themselves—they are thinking about the manipulatives.

Alternatively, they might use manipulatives to solve a task. For example, students might use Algebra Tiles (see Figure 1.2) to solve tasks such as

Figure 1.2 | Algebra Tiles

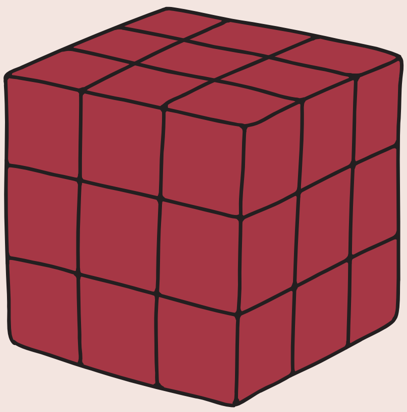


1. $(x + 2)(x + 3) =$
2. $(x + 4)(x - 3) =$
3. $(x - 2)(x - 4) =$

For these tasks, Algebra Tiles allow the students to use the area model with positive and negative terms to multiply binomials—they are thinking with the manipulatives.

A very particular form of thinking with manipulatives is for students to use manipulatives to help them *see* what the task is asking. For example, cubes (see Figure 1.3) are a good way to help students *see* the painted cube problem.

Figure 1.3 | Painted cube



A 3 × 3 × 3 cube is dipped in red paint and then left to dry. Once dry, the cube is taken apart into its 27 original smaller cubes. How many of those smaller cubes have red paint on 3 sides, 2 sides, 1 side, and 0 sides?

If students have access to cubes, they can build the 3 × 3 × 3 shape to help them *see* what is being asked. And the ability to take the 3 × 3 × 3 cube apart will help them to think about what the task is asking—they are thinking with the manipulatives.

The third way in which manipulatives are used with tasks is to help students represent and communicate their thinking. Solving tasks in groups requires copious amounts of communication, negotiation, and persistence. It is sometimes difficult for students to make themselves understood using only their words, and manipulatives can be used to mediate this process. For example, a student who is having difficulty explaining why 4×-3 is -12

might grab some Integer Disks to show their group the −3 can be represented as 3 negative disks. And 4 × −3 is just 4 groups of these 3 negative disks. Alternatively, a student who is having difficulty explaining why an octahedron is a Platonic solid may grab two square pyramids and put them base to base to show their groupmates that every vertex in an octahedron has the same number of faces meeting at it. In each of these cases, the student is not using manipulatives to think with—they have already done the thinking. Instead, they are using manipulatives to help their groupmates understand what they are trying to communicate—they are helping their groupmates to think with the manipulatives.

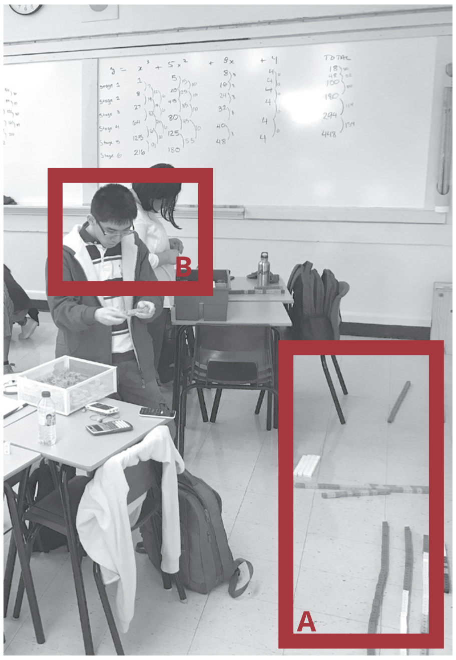
Regardless of the use, however, it is important that students have either explicit or implicit access to the manipulatives. Explicit access is where you, as the teacher, decree that they are to use the manipulatives, and maybe even use them in a specific way. Explicit access to manipulatives is needed when students are solving tasks that require them to think about the manipulative—like the previous Algebra Tiles tasks. Implicit access means, most often, that there are several different manipulatives available to students should they need them. Students who want to use manipulatives to communicate their thinking—like the previous integer or 3D-shape tasks—would then select which ones they want and go get them. To think with manipulatives—like the painted cube example—students can have either explicit or implicit access to manipulatives.

Whichever purpose manipulatives are being used for, it is important to remember that manipulatives are only one form of mathematical representation—one of five representations as seen in the Lesh Translation Model (Lesh et al., 1987):

1. Manipulatives
2. Pictures
3. Written symbols
4. Verbal symbols
5. Real life situations

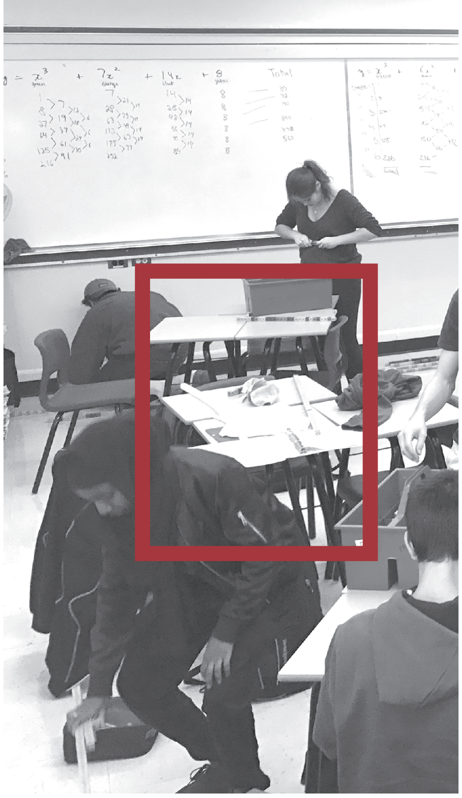
It is important that what students learn is how to translate between these different representations. One of the best ways to ensure that translation between these five forms of mathematical representation is to make sure that they are doing more than one at the same time. This means that while they are manipulating, we also need them to be talking and drawing and writing. A thinking task gives students something to talk about and visibly random groups give students someone to talk with. But where is the best workspace when students are using manipulatives? Traditionally, this kind of work, along with most student work, has been done with students sitting at their desks. BTC showed us that working on a task with the powerful effect of VNPSs was far superior to this and that turns out to be true when incorporating manipulatives as well. This is most easily achieved by having a group manipulate on the floor next to a VNPS (see A in Figure 1.4), on a ledge or shelf at the VNPS (see B in Figure 1.4), or on a desk or table pulled close to a VNPS (Figure 1.5). The manipulatives give the students models to think about, think with, and communicate with while the VNPS provides a workspace that has been proven to create better engagement, enthusiasm, perseverance, knowledge mobility, time on task—and thinking. Together they work to help students translate between representations.

Figure 1.4 | Manipulatives on the floor and on a ledge



Credit: Alex Overwijk, 2017

Figure 1.5 | Manipulatives on a desk near a VNPS



Credit: Alex Overwijk, 2017

There will be a temptation to have students work with manipulatives while sitting in their desks. Do not give into this temptation. Even if you have them working on a horizontal non-permanent surface like a whiteboard laying on a table or an erasable desk, the students are sitting. And when they are sitting, they feel anonymous, and they will disengage faster than if they are standing.

THE USE OF TECHNOLOGY WITH THINKING TASKS

As with manipulatives, technology should be ubiquitous within a thinking classroom. And as with manipulatives, there are three main uses of technology in a thinking classroom. Whether we are using calculators, Desmos, Geogebra, Polypad, or Scratch students use technology as

1. a computational tool
2. an exploratory tool, or
3. a coding tool.

The first of these involves students using technology to calculate results for expressions, to plot data, or to measure a geometric figure, all for the purposes of computing something. In the research, we have found that for tasks that require computation, one device—preferably handheld—in each group is sufficient. This promotes collaboration in the same way that one marker per VNPS promotes collaboration. If everyone had their own marker and/or their own device, students would drift away from each other rather than toward each other. We have also found that everything that goes into the device and comes out of the device must be written on the VNPS. The reason for this is twofold. First, students often make mistakes when keying something into a device. Writing down what they key in as well as the result that the technology gives makes it easier to spot input errors. Second, the person with the device can become separated from the group as they begin to do their own thing. By having the group, as a whole, decide what goes into the device, the technology becomes a communal tool rather than a tool that only one person uses.

Technology can also be an exploratory tool. For example, students can explore what happens to

- the graph of a parabola as they change the coefficients of a quadratic;
- the line segments connecting adjacent midpoints of the sides of a quadrilateral as they change the quadrilateral; and
- the mean, the median, and the mode as they change numbers in a data set.

All of these activities require students to work in a cycle of observing, conjecturing, and testing and technology provides them both with outcomes to think about and a tool to test conjectures. Like with technology as a computational tool, this works best when there is one device per group. And, although there is too much happening to record everything they do on the VNPS, this works best if they, at least, record their conjectures and the results of these conjectures. Having multiple data points on the VNPS is what will allow them to begin to see patterns.

Finally, technology can be used as a coding tool. Whether students are telling a turtle where they want it to go, programming a robot, directing a programmable car, or writing code to get a computer to complete some task, the technology is inseparable from the task—they are thinking about the technology. Coding, by its very nature, is exploratory. This is how we learn new functionality and how we debug our code, and both are achieved by making note of inputs and outputs. The difference between coding and the exploration I've described is that, while exploration is often constrained to one or two variables, coding is more comprehensive, requiring students to bring together multiple routines. The other difference is that coding is rarely done on handheld devices but on laptops and, as such, is difficult to do solely at a VNPS. In the research, we found that what worked best for coding was that students work in groups of two, have access both to a VNPS and a desk or table where they can sit, and have one device for every group. VNPSs can be used to plan out their code and to record some of their explorations.

It is important to remember that technology, like manipulatives do not make a task good. A bad task with manipulatives is still a bad task. Likewise, it doesn't matter how good the animations are or how many badges you get, technology does not make a bad task good. For a task to be good, it will still need to have a low floor, high ceiling, and novelty. In Chapter 4, we share how to achieve this through a sequence of tasks.

FAQ

Q What is the difference between a rich task and a non-curricular task?

A A rich task is any task that requires students to draw on a wide diversity of mathematical knowledge and to put this knowledge together in different ways in order to solve the problem. If some of this diverse knowledge happens to map onto the curriculum you are teaching, then the rich task can also be considered as a curricular task—but only for those students who happen to use this curricular knowledge. For students who do not, this rich task is a non-curricular task.

The similarity between non-curricular tasks and rich tasks lies not in what they are as much as what they do—and what they both do well is engage. Consider the open middle task earlier in this chapter:

Using the digits 1 to 9 at most one time each, place a digit in each box to create an expression with the greatest possible value.

$$\frac{\square^{\square} \times \square^{\square}}{\square^{\square} \times \square^{\square}} =$$

For a grade 8 student learning how to operate with exponents, this is a curricular task. For everyone else, this is a non-curricular task. Regardless, this is a rich task with the potential to tap into a variety of different ways to think about and represent numbers, to use exponents, and to think about how bases and exponents can be used to make large and small numbers. And, for most, it is a highly engaging task that can be used in the early days of building a thinking classroom.

Q Are all manipulatives good?

A Manipulatives are inert. They are neither good nor bad. What makes a manipulative good or bad is how they are used and for what purpose. Manipulatives, at their core, are a model to help mediate a connection between concrete and abstract ways of learning. Manipulatives that can be used to do this well are good manipulatives in that context and for that purpose. Algebra tiles are great at helping students multiply binomials. Unifix cubes could not be used to achieve the same outcome to the same effect. However, Unifix cubes are just as effective at helping students understand that 4×-3 is -12 as integer disks.

Q You use the word tool when talking about technology but not when talking about manipulatives. Why is that?

A For us, a tool is something that you will always use to achieve a task. Just like a hammer will always be used to drive a nail, a calculator will always be used to perform arithmetic with very large or very small numbers. A calculator is a tool—you will always use it. Manipulatives are models. A model is useful for students to get from point A to point B, but with time, they will move past them. Consider, for example, Algebra Tiles. Algebra Tiles are great at helping students multiply binomials. But a student will eventually move beyond the model and onto more symbolic ways of doing this. Tools stay with us. As we acquire more abstract ways of thinking, we leave our models behind us.

Q If some members of a group are using manipulatives on the floor or a desk and one member is standing at the board, won't the group become fractured?

A Yes—but only if there is no communication between group members. The same can be said for having only one marker—one student is engaged in writing, the other two are not. The fact that one is writing while others manipulate can drive really good multi- and intra-representational conversation. But just like it is with one marker, the roles should switch and switch frequently.

Q I don't like using manipulatives because the students just play with them rather than using them how I want. The same is true of technology. How do I prevent this?

A Don't prevent it. Trying to get students to not play is like trying to hold back the ocean with a broom. Know that whenever you give students a new manipulative or a new technology, they need 20 minutes to get the yaya's out. They need to play. Let them. If you do not, they will anyways. But you can use this to your advantage. Let them play and then vector that play as the means through which they become familiar with the manipulative and/or technology.

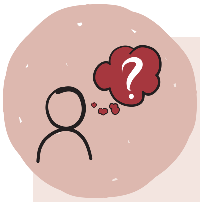
Q You talk about the use of manipulatives as something they can think with. The example you gave was blocks to help students think about the painted cube problem. Wouldn't they think more if we didn't give them a manipulative?

A Students are still thinking even with the manipulative. What they are not doing, however, is *visualizing.* Visualizing is what we do when we don't have a visual.

If we have a visual, we tend not to visualize. So, if the type of thinking you want students to do is visualize, then do not give them a visual—whether that comes in the form of a manipulative or technology. To be clear, we are not saying to not use visuals. Use them. And use them a lot. But recognize that if you want to develop the skill of visualization, you need to withhold the visual from time to time.

> Visualizing is what we do when we don't have a visual.

Alternatively, when you do the painted cube task, for example, you can show them a $3 \times 3 \times 3$ cube. Use it to explain the task. And then put it away. Then, while the students are working on the task at their whiteboards, you can carry the cube with you and let groups use it to explain their thinking. But then you take it away. And if there is a group that is really struggling to visualize, you would leave the cube with them for a while. In this fashion, you are using the visual to support visualizing rather than replace visualizing.



Questions to Think About

1. What in this chapter feels immediately correct?
2. What in this chapter challenges you?
3. Think about a task that you have used in the past that really got students to think. Reflect on this task through the lens of some of the things you learned in this chapter. What were the things about the task that made it good?
4. In this chapter you learned about the different purposes for manipulatives. Think about the last time you used manipulatives in your classroom. Which purpose were you using them for?
5. In this chapter you also learned about the three uses of technology. Think about the last time you used technology in your classroom. What were you using it for?
6. Think about a task that you think will be made better with the use of manipulatives and/or technology. What is it about this task that makes it so it will get better with the use of manipulatives and/or technology?
7. Now think about a task that would not be made better with the use of manipulatives and/or technology. What is it about this task that makes it so it will not get better with the use of manipulatives and/or technology?